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Field theories from physical requirements: Noether's first theorem, energy-momentum tensors and the question of uniqueness

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Abstract

An axiomatic approach to physics is proposed for obtaining classical gauge theories from a common set of physical requirements based on standard features of special relativistic field theories such as gauge invariance, conformal invariance and being in four dimensions. This approach involves the use of Noether's first theorem to directly obtain a unique, complete set of equations from the symmetries of the action. However, implementation of this procedure is obstructed by issues of ambiguity and non-uniqueness associated with the conserved tensors in the majority of special relativistic field theories. In the introductory chapter, we outline the three major problems which are considered in this thesis. Each of these three problems are addressed separately in the three central chapters of the thesis, which consist of eight integrated articles. These three problems are (i) the failure of the canonical Noether energy-momentum tensor to obtain known physical conservation laws, and the ad-hoc "improvement" of the energy-momentum tensors occurring in the literature, (ii) the ambiguities and non-uniqueness associated with multiple different methods for derivation of the energy-momentum tensor, and (iii) the procedure required for converting a set of axioms to a set of Lagrangian densities. The concluding chapter summarizes our major results, such as proper variational "Noetherian" symmetries for several completely gauge invariant models using the Bessel-Hagen method, a formal disproof of the equivalence of the Noether and Hilbert energy-momentum tensors in Minkowski spacetime, a proof that there are infinitely many solutions for energy-momentum tensors in linearized gravity obtained from the "improvement" method, and a derivation of the curvature tensors of higher spin gauge theories without referring to the symmetry properties of the Riemann curvature tensor. Future research that could follow from our results is discussed.

Keywords: Noether's first theorem, Bessel-Hagen method, energy-momentum tensor, gauge theory, electrodynamics, Gauss-Bonnet gravity, spin-2, higher spin gauge theories

Summary for Lay Audience

Physics is a science which focuses on quantifying observed natural phenomena. To do this for a classical system, physicists use equations that can be solved, subject to initial conditions corresponding to the dynamics of a particular observed phenomena; these equations are known as equations of motion. For the fundamental interactions of electromagnetism and gravity, the accepted equations of motion describing the dynamics of these theories are Maxwell's equations and Einstein's field equations, respectively. These equations have not been replaced or changed in over 100 years (although some modifications have been proposed). Other equations may be needed to complete a theory, such as in electrodynamics where conservation of energy and the force law are described by Poynting's theorem and the Lorentz force law. Using what is known as the action, the equations of motion of the theory can be straightforwardly obtained using the Euler-Lagrange equation; this equation ensures that this action has a minimum value. For conservation laws however, this is not as straightforward — multiple methods which contradict each other exist for obtaining them. In addition, some of these methods fail to obtain all known physical laws in a straightforward manner. These issues are the focus of the first two chapters of this thesis. The basis for our approach is Noether's first theorem, a fundamental result that shows that symmetries present in a physical system results in there being "conserved" quantities (quantities whose value remains constant as the system evolves). In Chapter 1 we clarify a straightforward methodology for obtaining physical conservation laws using the Bessel-Hagen approach to Noether's first theorem, and then in Chapter 2 we clarify the status of the other methods that contradict this approach. This use of Noether's first theorem renders the complete set of physical equations, for example in electrodynamics, as implicit information contained in the Lagrangian. For a set of physical theories, only the set of Lagrangians are required. In Chapter 3 we ask if all of these Lagrangians can be obtained from a common set of axioms, so that even the set of Lagrangians are implicit information to the imposed physical requirements. We show that such an approach can be used to obtain electrodynamics, linearized Gauss-Bonnet gravity, and other "higher-spin" gauge theories found in the physics literature. Future research that could follow from our results is discussed.

Co-Authorship Statement

In total there are eight articles included in this integrated article PhD thesis [15, 12, 13, 10, 8, 14, 9, 11] (appearing in this order). Six of these articles have been published or accepted for publication, in journals indicated below. Authorship for each will be discussed below, with co-authored papers given a detail of the breakdown of duties between the following four categories: conceptualization, writing first draft, writing final draft, and formal analysis.

Starting with the single authored papers [10, 8, 9]. All of these are authored by the author of the thesis, Mark Robert Baker. All four categories (conceptualization, writing first draft, writing final draft, and formal analysis) were completed by this author (Baker). The paper [8] was published in the journal *Classical and Quantum Gravity*. The paper [9] was published in *International Jouranl of Modern Physics D*.

The article [15] in Section 2.1 of Chapter 2 was co-authored by Mark Robert Baker, Niels Linnemann and Chris Smeenk. Duties breakdown was: conceptualization (Baker), writing first draft (Baker, Linnemann, Smeenk), writing final draft (Baker, Linnemann, Smeenk) and formal analysis (Baker). This article is to appear in the collected papers volume "*The Physics and Philosophy of Noether's Theorems*" published by Cambridge University Press.

The article [12] in Section 2.2 of Chapter 2 was co-authored by Mark Robert Baker, Natalia Kiriushcheva and Sergei Kuzmin. Duties breakdown was: conceptualization (Baker and Kuzmin), writing first draft (Kuzmin), writing final draft (Baker, Kiriushcheva and Kuzmin) and formal analysis (Baker, Kuzmin and Kiriushcheva).

The article [13] in Section 3.1 of Chapter 3 was co-authored by Mark Robert Baker, Natalia Kiriushcheva and Sergei Kuzmin. Duties breakdown was: conceptualization (Baker), writing first draft (Baker), writing final draft (Baker, Kiriushcheva and Kuzmin) and formal analysis (Baker, Kiriushcheva and Kuzmin). This article was published in the journal *Nuclear Physics B*.

The article [14] in Section 4.1 of Chapter 4 was co-authored by Mark Robert Baker and Sergei Kuzmin. Duties breakdown was: conceptualization (Baker and Kuzmin), writing first draft (Baker), writing final draft (Baker and Kuzmin) and formal analysis (Baker and Kuzmin). This article was published in the journal *International Journal of Modern Physics D*.

The article [11] in Section 4.3 of Chapter 4 was co-authored by Mark Robert Baker and Julia Bruce-Robertson. Duties breakdown was: conceptualization (Baker), writing first draft (Baker), writing final draft (Baker and Bruce-Robertson) and formal analysis (Baker and Bruce-Robertson). This article was published in the journal *Canadian Journal of Physics*.

Epigraph

"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

There are at present fundamental problems in theoretical physics awaiting solution, e.g., the relativistic formulation of quantum mechanics and the nature of atomic nuclei (to be followed by more difficult ones such as the problem of life), the solution of which problems will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence to get the necessary new ideas by direct attempts to formulate the experimental data in mathematical terms. The theoretical worker in the future will therefore have to proceed in an indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formulation that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities (by a process like Eddington's Principle of Identification)."

Paul Dirac, 1931 [64]

Dedication

Dedicated to physics, asking questions, and the freedom to do both.

Acknowlegements

This thesis was made possible in large part thanks to the support of numerous individuals and groups throughout three degrees (BSc, MSc, PhD) completed at the University of Western Ontario in Physics, Applied Mathematics and at the Rotman Institute of Philosophy. These departments and institute have been my home for a decade now and I could not have picked a better place for my studies and growth as a physicist; the way they supported me and allowed me to grow and develop is something that I appreciated every day — to all those involved, past and present, thank you. I truly feel my research (and to some degree, myself) is at the "triple point" of physics, mathematics and philosophy. The remainder of this acknowledgements section hopes to say thanks to some of the individuals who helped make this thesis possible.

First to my family; my parents, brother and grandparents — the endless support I have received throughout my life has been paramount in allowing me to pursue my dream of studying theoretical physics. If not for their encouragement, support and security it is possible that I would not have been able to relentlessly pursue exactly the research I truly believe in. It is possible that without them I would have ended up "working" or doing research for the wrong reasons (money, career, etc.). Such alternative paths would have rendered me a fraction of the physicist and person that I am today. A soon to be new addition to my family, is my fiancé Joy, as we will be getting married shortly after the defense of this thesis. In many ways she is already a part of my family and I could not be happier to have her in my life. I cannot imagine going on the remainder of my journey as a theoretical physicist without her, and in many ways, this will be her journey too. I assure the reader that as a high school student she attended the Perimeter Institute for Theoretical Physics and began her undergraduate degree in Physics; i.e. she has a more than sufficient appreciation for physics required to embark on this journey.

My co-workers, research colleagues and supervisors have made a great impact on my research and growth. Co-authors Niels, Chris and Julia that appear in this thesis, as well as advisory committee members Wayne and Aaron, were all invaluable members of this development. My PhD supervisor Gerry has been all I could ever ask for in a PhD supervisor; he allowed me to grow in the most organic way possible, providing guidance and wisdom without ever imposing his direction on me. Natalia, the wife of my co-supervisor has provided me with countless instruction on the mathematics I use every day; she deserves credit as both the best mathematician in our research group and as an unofficial "third" supervisor. Finally, Sergei, my co-supervisor, is the one who showed me that theoretical physics, as I had dreamed as a young student, does in fact exist. Often when students are asked who is their idol physicist, they respond with Newton, or Maxwell, or Einstein, or Noether. For me, I get to work with my idol physicist every day. I only hope that he, my other supervisors, and colleagues at Western, can be proud of the physicist that I have become.

Contents

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List of Abbreviations, Symbols and Nomenclature

- $\mu, \nu, \alpha, \beta, \gamma, \ldots$ — Greek indices represent 4 dimensions (1,2,3,4)
- $\vec{E} =< E_x, E_y, E_z >$ — Electric field vector
- $\vec{B} =< B_x, B_y, B_z >$ — Magnetic field vector
- $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} =< S_x, S_y, S_z >$ — Poynting vector
- $J^\sigma =< c\rho, J_x, J_y, J_z >$ — Four current
- $\vec{\nabla} =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$ — del operator
- $\partial_\mu =< \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >=< \frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z >$ — Covariant partial derivative
- $\partial^\mu =< \frac{1}{c}\partial_t, -\partial_x, -\partial_y, -\partial_z >$ — Contravariant partial derivative
- $\Box = \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$ — d'Alembertian operator
- $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ — Relation between dual and non-dual field strength
- $\delta^\nu_\mu = g_{\alpha\mu}g^{\alpha\nu}$ — Kronecker delta from contracted metrics
- $\delta^\gamma_\zeta = \frac{\partial \bar{x}^\alpha}{\partial x^\zeta}\frac{\partial x^\gamma}{\partial \bar{x}^\alpha}$ — Kronecker delta from transformation inverses
- $v_i = \frac{\partial \bar{x}^p}{\partial x^i}\bar{v}_p$ — Covariant vector (first rank covariant tensor)
- $g_{ij} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\bar{g}_{pq}$ — Covariant tensor (second rank covariant tensor)
- $\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^p}v^p$ — Contravariant vector (first rank contravariant tensor)
- $\bar{g}^{ij} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}g^{pq}$ — Contravariant tensor (second rank contravariant tensor)
- $\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha{}_{\nu\mu}v^\nu$ — Covariant derivative of a contravariant vector
- $\nabla_\mu v_\alpha = \partial_\mu v_\alpha - \Gamma^\lambda{}_{\mu\alpha}v_\lambda$ — Covariant derivative of a covariant vector
- $\nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha{}_{\gamma\lambda}T^{\lambda\beta} + \Gamma^\beta{}_{\gamma\lambda}T^{\alpha\lambda}$ — Covariant derivative of a second rank contravariant tensor
- $\nabla_\gamma T_{\alpha\beta} = \partial_\gamma T_{\alpha\beta} - \Gamma^\lambda{}_{\gamma\alpha}T_{\lambda\beta} - \Gamma^\lambda{}_{\gamma\beta}T_{\alpha\lambda}$ — Covariant derivative of a second rank covariant tensor

- $T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}$ — Hilbert energy-momentum tensor in curved spacetime
- $T_{H,\eta}^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}\Big|_{g=\eta}$ — Hilbert energy-momentum tensor in Minkowski spacetime
- $\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}} = \frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} - \partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} + \partial_\xi\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} + \ldots$ — Euler derivative with respect to the metric
- $\frac{\partial\sqrt{-g}}{\partial g_{\gamma\rho}} = \frac{1}{2}g^{\gamma\rho}\sqrt{-g}$ — Derivative of square root of -g
- $\frac{\partial g^{\lambda\nu}}{\partial g_{\beta\gamma}} = -\frac{1}{2}(g^{\beta\lambda}g^{\gamma\nu} + g^{\gamma\lambda}g^{\beta\nu})$ — Derivative of contravariant metric with respect to covariant metric
- $R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha})$ — Linearized Riemann tensor
- $R^{\nu\beta} = \eta_{\mu\alpha}R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\beta\partial^\alpha h^\nu_\alpha + \partial^\nu\partial^\alpha h^\beta_\alpha - \Box h^{\nu\beta} - \partial^\nu\partial^\beta h)$ — Linearized Ricci tensor
- $R = \eta_{\nu\beta}R^{\nu\beta} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h$ — Linearized Ricci scalar
- $R^\rho{}_{\beta\alpha\gamma} = \partial_\alpha\Gamma^\rho{}_{\gamma\beta} - \partial_\gamma\Gamma^\rho{}_{\alpha\beta} + \Gamma^\lambda{}_{\gamma\beta}\Gamma^\rho{}_{\alpha\lambda} - \Gamma^\lambda{}_{\alpha\beta}\Gamma^\rho{}_{\gamma\lambda}$ — Riemann tensor
- $R_{\mu\beta\alpha\gamma} = \frac{1}{2}(\partial_\alpha\partial_\beta g_{\mu\gamma} + \partial_\gamma\partial_\mu g_{\alpha\beta} - \partial_\alpha\partial_\mu g_{\gamma\beta} - \partial_\gamma\partial_\beta g_{\mu\alpha}) + g_{\rho\lambda}(\Gamma^\lambda{}_{\mu\gamma}\Gamma^\rho{}_{\alpha\beta} - \Gamma^\lambda{}_{\mu\alpha}\Gamma^\rho{}_{\gamma\beta})$ — Covariant Riemann tensor
- $R_{\mu\beta\alpha\gamma} = -R_{\beta\mu\alpha\gamma} \quad , \quad R_{\mu\beta\alpha\gamma} = -R_{\mu\beta\gamma\alpha}$ — Antisymmetry of each pair in covariant Riemann and linearized Riemann tensors
- $R_{\mu\beta\alpha\gamma} = R_{\alpha\gamma\mu\beta}$ — Symmetry of pair interchange in covariant Riemann and linearized Riemann tensors
- $(\nabla_\gamma\nabla_\alpha - \nabla_\alpha\nabla_\gamma)v_\beta = R^\rho{}_{\beta\alpha\gamma}v_\rho$ — Riemann tensor from non-commutivity of covariant derivatives
- $R_{\mu\alpha\beta\gamma} + R_{\mu\gamma\alpha\beta} + R_{\mu\beta\gamma\alpha} = 0$ — First Bianchi identity
- $\Gamma_{\lambda\mu\alpha} = \frac{1}{2}(-\partial_\lambda h_{\mu\alpha} + \partial_\alpha h_{\lambda\mu} + \partial_\mu h_{\lambda\alpha})$ — linearized Christoffel symbol
- $\Gamma_{\alpha\rho\sigma} = \frac{1}{2}(\partial_\sigma g_{\alpha\rho} + \partial_\rho g_{\alpha\sigma} - \partial_\alpha g_{\rho\sigma})$ — Christoffel symbol of the first kind
- $\Gamma^\lambda_{\nu\beta} = \frac{1}{2}g^{\mu\lambda}(-\partial_\mu g_{\nu\beta} + \partial_\beta g_{\mu\nu} + \partial_\nu g_{\mu\beta})$ — Christoffel symbol of the second kind
- $\bar{\Gamma}_{\alpha\rho\sigma} = \frac{\partial x^\gamma}{\partial\bar{x}^\alpha}\frac{\partial x^\beta}{\partial\bar{x}^\rho}\frac{\partial x^\lambda}{\partial\bar{x}^\sigma}\Gamma_{\gamma\beta\lambda} + g_{\gamma\beta}\frac{\partial x^\gamma}{\partial\bar{x}^\alpha}\bar{\partial}_\sigma\frac{\partial x^\beta}{\partial\bar{x}^\rho}$ — Transformation law for Christoffel symbol of the first kind
- $\bar{\Gamma}^\mu{}_{\rho\sigma} = \frac{\partial\bar{x}^\mu}{\partial x^\xi}\frac{\partial x^\beta}{\partial\bar{x}^\rho}\frac{\partial x^\lambda}{\partial\bar{x}^\sigma}\Gamma^\xi{}_{\beta\lambda} + \frac{\partial\bar{x}^\mu}{\partial x^\xi}\bar{\partial}_\sigma\frac{\partial x^\xi}{\partial\bar{x}^\rho}$ — Transformation law for Christoffel symbol of the second kind

- $\partial_\sigma g_{\alpha\rho} = \Gamma_{\alpha\rho\sigma} + \Gamma_{\rho\alpha\sigma}$ — Derivative of metric tensor

- $\nabla_\sigma g_{\alpha\rho} = 0$ — Covariant derivative of metric tensor

- $\delta x_\alpha = a_\alpha + \omega_{\alpha\beta}x^\beta + S\, x_\alpha + 2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu$ — Conformal transformations

- $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ — Spin-2 gauge transformation (linearized diffeomorphism)

- $\Delta^{\gamma\rho}_{\nu\alpha} = \frac{1}{2}(\delta^\gamma_\nu \delta^\rho_\alpha + \delta^\gamma_\alpha \delta^\rho_\nu)$ — Symmetric variation

- $T^{\mu\nu} = \begin{pmatrix} -U & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ -S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ -S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$ — Energy-momentum tensor of electrodynamics

- $\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$ — Maxwell stress tensor

- $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ — Minkowski metric

- $g_{\mu\nu} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$ — Metric tensor

- $\delta^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ — Kronecker delta

- $\frac{\partial \bar{x}^\alpha}{\partial x^\mu} = \Lambda^\alpha_\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ — Lorentz boost in x direction

- $F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$ — Covariant field strength tensor of electrodynamics
- $F^{\rho}{}_{\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$ — Mixed field strength tensor of electrodynamics
- $F^{\rho\sigma} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$ — Contravariant field strength tensor of electrodynamics
- $\mathcal{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$ — Covariant dual field strength tensor of electrodynamics
- $\mathcal{F}^{\rho}{}_{\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$ — Mixed dual field strength tensor of electrodynamics
- $\mathcal{F}^{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$ — Contravariant dual field strength tensor of electrodynamics

Chapter 1

Introduction

1.1 Electrodynamic theory

1.1.1 Maxwell's equations

Over the past centuries since Isaac Newton's "Philosophiæ Naturalis Principia Mathematica" [158], physics has been vastly successful in quantifying the dynamics of observed phenomena. Equations of motion that model the most fundamental interactions have been some of the major focuses of theoretical physics during this period. The prototypical example of such a model is Maxwell's equations, which unified the theories of electricity and magnetism in 1861 [145]. Maxwell originally presented these equations as 8 separate equations, which correspond to the 8 equations found by expanding the more common form presented by Heaviside in 1894 [102] (in the vector notation introduced by Gibbs [90]),

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}, \tag{1.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.2}$$

where the four nonhomogenous equations (1.1) are the Gauss-Ampere laws, and the four homogenous equations (1.2) are the Gauss-Faraday laws. The four nonhomogenous equations (1.1) are sourced by charge density ρ and current density $\vec{J} =< J_x, J_y, J_z >$.

In 1909 [150], Minkowski formulated the 4D metric spacetime of special relativity now known as the Minkowski metric $\eta^{\mu\nu}$, where $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$ is the invariant interval of special relativity, a special case of the general line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Here he further compacted Maxwell's equations into the divergence of what we now call the field strength $F^{\mu\nu}$ and dual field strength $\mathcal{F}^{\mu\nu}$ tensors,

$$\partial_\rho F^{\rho\sigma} = \mu_0 J^\sigma, \tag{1.3}$$

$$\partial_\rho \mathcal{F}^{\rho\sigma} = 0. \tag{1.4}$$

See List of Symbols for the components of the field strength and dual field strength tensors — the components of these tensors are the electric and magnetic fields. The four Gauss-Ampere equations (the nonhomogeneous Maxwell equations) are compactly expressed in (1.3) using the field strength tensor, with each component of the 4-vector $\partial_\mu F^{\mu\nu}$ corresponding to one of these four equations. The four Gauss-Faraday equations (the homogenous Maxwell equations) are compactly expressed in (1.4) using the dual field strength tensor, with each component of the 4-vector $\partial_\mu \mathcal{F}^{\mu\nu}$ corresponding to one of these four equations. The more common presentation of covariant electrodynamics is to write all 8 equations in terms of just the field strength tensor $F^{\mu\nu}$,

$$\partial_\rho F^{\rho\sigma} = \mu_0 J^\sigma, \tag{1.5}$$

$$\partial_\sigma \mathcal{F}^{\sigma\rho} = \frac{1}{6}\epsilon^{\alpha\beta\sigma\rho}(\partial_\sigma F_{\alpha\beta} + \partial_\beta F_{\sigma\alpha} + \partial_\alpha F_{\beta\sigma}) = 0. \tag{1.6}$$

This presentation dates back to Einstein [69] which is possible due to the relationships $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. More commonly the Gauss-Faraday law (1.6) is presented simply as the Bianchi identity $\partial_\sigma F_{\alpha\beta} + \partial_\beta F_{\sigma\alpha} + \partial_\alpha F_{\beta\sigma} = 0$, in the brackets of (1.6). We discuss the dual formulation at length in Section 3 of [9].

1.1.2 Lorentz force and conservation laws

The theory of electricity and magnetism, more commonly referred to as classical electrodynamics, consists of the complete set of equations that describe observed electrodynamic phenomena. Maxwell's equations, the equations of motion for the theory, are just one part of this complete set. The Lorentz force law defines the force electromagnetic fields exert on charged particles,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{1.7}$$

The component form of this expression can be traced back to Maxwell [145] but was named after Lorentz for his presentation in [136]. Einstein derived this in his original paper on special relativity using the Lorentz transformations [68]. We note that (1.7) can be obtained from the classical Lagrangian for a charged particle in an electromagnetic field using the Euler-Lagrange equation. In covariant electrodynamics this law can also be obtained from the compactly expressed force density, in terms of the field strength tensor,

$$f^\nu = F^\nu{}_\alpha J^\alpha, \tag{1.8}$$

where J^α is the four-current (source of the nonhomogeneous Maxwell equations). Specifically the Lorentz force density is the spatial components of this force density $f^i = J^\rho F^i{}_\rho = \rho\vec{E}+\vec{J}\times\vec{B}$. There are also several conservation laws, such as Poynting's theorem,

$$-\frac{\partial U}{\partial t} = \vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E}, \tag{1.9}$$

where $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ is the Poynting vector and $U = \frac{1}{2}(E^2+B^2)$ is the energy density of the fields. This law and all conservation laws of electrodynamics have a compact covariant form. In the case of Poynting's theorem the law is expressed as the divergence of the energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = \frac{1}{\mu_0}[F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}]. \tag{1.10}$$

This energy-momentum tensor describes the flow of energy and momentum of the electric and magnetic fields. Its components include the energy density of fields U, the Poynting vector $\vec{S}$ and the Maxwell stress tensor σ^{ji}, first published in [146]. See List of Symbols for details.

From the divergence of this we have the on-shell conservation of energy and momentum laws for electrodynamics theory,

$$\partial_\mu T^{\mu\nu} = \frac{1}{\mu_0}[(\partial_\mu F^{\mu\rho})F^\nu{}_\rho + \frac{1}{2}F_{\sigma\rho}(\partial^\rho F^{\sigma\nu} + \partial^\sigma F^{\nu\rho} + \partial^\nu F^{\rho\sigma})] = 0, \tag{1.11}$$

where by on-shell we mean that using the complete set of Maxwell's equations (all 8) and setting them equal to zero, then $\partial_\mu T^{\mu\nu} = 0$. If instead we replace $\partial_\mu F^{\mu\rho} = J^\rho$ with the four current, as we will soon see, we recover the Lorentz force density.

Specifically, Poynting's theorem (1.9) is recovered from the time component $\partial_\mu T^{\mu 1} = 0$. Poynting's theorem is just one of fifteen conservation laws of the electromagnetic fields. Four are included in $\partial_\mu T^{\mu\nu}$, the three spatial components being the conservation of the Maxwell stress tensor,

$$\partial_\mu T^{\mu i} = -\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} + \partial_j\sigma^{ji} = 0. \tag{1.12}$$

The remaining eleven conserved tensors can be expressed in terms of the field strength tensor arranged as the energy-momentum tensor (1.10). Six are compactly expressed in terms of the divergence of the angular momentum tensor $M^{\rho\alpha\beta}$ [112],

$$M^{\rho\alpha\beta} = x^{\alpha}T^{\rho\beta} - x^{\beta}T^{\rho\alpha}, \tag{1.13}$$

where $\partial_{\rho}M^{\rho\alpha\beta}$ has 6 independent components (due to the antisymmetry in $[\alpha\beta]$) representing the 6 angular momentum conservation laws of the electromagnetic fields. This tensor is conserved on-shell using the symmetry property of the energy-momentum tensor $\partial_{\rho}M^{\rho\alpha\beta} = \delta^{\alpha}_{\rho}T^{\rho\beta} - \delta^{\beta}_{\rho}T^{\rho\alpha} = 0$. Four laws associated to the conformal tensor $C^{\rho\alpha}$ [26],

$$C^{\rho\alpha} = T^{\rho\beta}(2x_{\beta}x^{\alpha} - \delta^{\alpha}_{\beta}x_{\lambda}x^{\lambda}), \tag{1.14}$$

where $\partial_{\rho}C^{\rho\alpha}$ has 4 independent components associated to the 4 conformal conservation laws. This tensor is conserved on-shell as a consequence of the traceless energy-momentum tensor $\partial_{\rho}C^{\rho\alpha} = 2T^{\beta}_{\beta}x^{\alpha} + 2T^{\rho\beta}(x_{\beta}\delta^{\alpha}_{\rho} - \delta^{\alpha}_{\beta}x_{\rho}) = 0$. The final (fifteenth) law is associated to the dilatation tensor D^{ρ} [26],

$$D^{\rho} = T^{\rho\beta}x_{\beta}, \tag{1.15}$$

where $\partial_{\rho}D^{\rho}$ has one component associated to the dilatation conservation law, which is also conserved on-shell as a consequence of the traceless energy-momentum tensor $\partial_{\rho}D^{\rho} = T^{\rho}_{\rho} = 0$.

Altogether we have Maxwell's equations, the Lorentz force law, and the 15 conservation laws of electrodynamic theory. The Lorentz force law can be derived from the divergence of the energy-momentum tensor using the nonhomogenous equations of motion (four current on-shell condition),

$$f^{\nu} = \partial_{\mu}T^{\mu\nu} = F^{\nu}{}_{\alpha}J^{\alpha}. \tag{1.16}$$

Since the Lorentz force density is the spatial components of this force density it is therefore the spatial components of the divergence of the energy-momentum tensor after using the nonhomogenous on-shell condition $f^{i} = \partial_{\mu}T^{\mu i} = J^{\rho}F^{i}{}_{\rho} = \rho\vec{E} + \vec{J}\times\vec{B}$. Therefore, the equations electrodynamic theory can be compactly obtained from Maxwell's equations and the conservation laws alone. These equations, which are elegantly and compactly related by Noether's first theorem, are the focus of Section 1.3.3.

1.1.3 Potential formulation

To finish our summary of electrodynamics we turn to the potential formulation. Recall the electric and magnetic fields can be expressed in terms of potentials,

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \tag{1.17}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{1.18}$$

where $\vec{A}$ is the magnetic vector potential $\vec{A} =< A_x, A_y, A_z >$ and ϕ is the electric scalar potential. Together these potentials are expressed as the four potential $A^\mu =< \frac{1}{c}\phi, A_x, A_y, A_z >$. The field strength tensor we introduced in the previous sections can be compactly expressed using the four potential as,

$$F^{\rho\sigma} = \partial^\rho A^\sigma - \partial^\sigma A^\rho, \tag{1.19}$$

where $F^{\rho\sigma}$ is antisymmetric in $[\rho\sigma]$ thus $F^{\rho\sigma} = -F^{\sigma\rho}$. The complete set of equations for electrodynamic theory can be compactly expressed in terms of this field strength tensor. An invariance property of this F is therefore shared by the complete set of equations. The invariance property at the heart of gauge theory, gauge invariance, is central to the motivations and arguments found throughout contemporary physics, this thesis included. In the case of electrodynamics, $F^{\rho\sigma}$ is invariant under the gauge transformation,

$$A^\mu \to A^\mu + \partial^\mu \chi, \tag{1.20}$$

where χ is the scalar gauge parameter. Thus the complete set of equations defining electrodynamics theory are also gauge invariant under (1.20). Gauge invariance is a property that allows for a freedom in potential of a gauge theory that can be exploited for practical purposes such as solving the wave equation in electrodynamics, where we expand the nonhomogenous Maxwell's equations in terms of the field strength tensor,

$$\partial_\rho \partial^\rho A^\sigma - \partial^\sigma \partial_\rho A^\rho = \mu_0 J^\sigma, \tag{1.21}$$

and use the Lorenz gauge fixing to eliminate the second term,

$$\partial_\rho A^\rho = \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0, \tag{1.22}$$

leaving only the wave equation $\Box A^\sigma = \mu_0 J^\sigma$. In gauge theory these potentials (often referred to simply as fields) are the dependent variables of the Lagrangian density used to write down the action of a given field theory. We will focus on analytical mechanics and Noether's first theorem in the following section. For now, we will simply state the Lagrangian density of electrodynamics,

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}. \tag{1.23}$$

Substituting this into the Euler-Lagrange equation yields the nonhomogeneous half of Maxwell's equations (see [9] for discussion of the other half analytically). This Lagrangian, built from $F_{\mu\nu}$, is also exactly gauge invariant.

1.2 Complete gauge invariance

The Lagrangian density, equations of motion and conservation laws of electrodynamics are all independently and exactly gauge invariant. This is a special property of a field theory which we refer to as complete gauge invariance throughout the thesis, as described in [14].

Complete gauge invariance is a desired property if one is to use gauge fixing/ impose a gauge transformation on one of the equations of a theory. This way gauge fixing, for example, the equation of motion will not change the Lagrangian density, energy-momentum tensor, etc. The property of complete gauge invariance is not one shared by all field theories which are considered to be gauge invariant. The best example is spin-2 theory [74, 147], where the spin-2 equation of motion (linearized Einstein's field equations),

$$E^{\mu\nu} = \frac{1}{2}[-\eta^{\mu\nu}\Box h + \Box h^{\mu\nu} + \partial^\mu\partial^\nu h - \partial_\lambda\partial^\nu h^{\mu\lambda} - \partial_\lambda\partial^\mu h^{\nu\lambda} + \eta^{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}], \tag{1.24}$$

is invariant under the spin-2 (linearized diffeomorphisms) gauge transformation,

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu. \tag{1.25}$$

However, the spin-2 (Fierz-Pauli) action,

$$\mathcal{L}_{FP} = \frac{1}{4}[\partial_\alpha h^\beta_\beta \partial^\alpha h^\gamma_\gamma - \partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} + 2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha} - 2\partial^\alpha h^\beta_\beta \partial^\gamma h_{\gamma\alpha}], \tag{1.26}$$

is not exactly invariant under this transformation, it is only invariant up to a boundary term [14]. Common practice is to consider any action which is invariant up to a boundary term as a gauge invariant theory; this is because by using Stokes' theorem one discards any boundary terms and these boundary terms do not impact the Euler-Lagrange equations of motion. The common view of a gauge invariant theory is therefore one which has a gauge invariant equation of motion. This view is problematic if we are to consider complete sets of equations for a given model.

In spin-2, the energy-momentum tensor is problematic, discussed at length in [8]. For starters, there are numerous published expressions in the literature for $T^{\mu\nu}$ of spin-2 [28], none of which are gauge invariant [140], thus it is not clear which should be considered the fundamental and unique expression to define the conservation laws of the theory. Magnano and Sokolowski [140] go so far as to say:

> "*Applying a physically undeniable condition that the energy–momentum tensor should have the same gauge invariance as the field equations, we also conclude that this approach to gravity does not furnish a physically acceptable notion of gravitational energy density.*"

Others claim that in spin-2 Fierz-Pauli theory, an energy-momentum tensor is required for certain calculations [163], for which there is still no consensus on which to choose. Regardless of interpretation, the fact remains that the numerous energy-momentum tensors for spin-2 Fierz-Pauli theory are all gauge dependent (see [8] for discussion of these expressions). Therefore, whichever we choose, values of energy-momentum and conservation laws of the field depend on the choice of gauge. Common practice in the literature is to solve the spin-2 equation of motion by selecting the de Donder gauge $\partial_\mu h^{\mu\nu} = 0$ [47], similar to using the Lorenz gauge to solve for wave equation in electrodynamics [112]. However, if we do this in spin-2 we change our Lagrangian density and conservation laws, since they are not gauge invariant. The Lagrangian case one can argue to neglect the boundary term, but the energy-momentum tensor has no such freedom. This fundamental difference with electrodynamics (outlined in the figure below),

Expression	**Symbol**	**Electrodynamics**	**Spin-2**
Lagrangian	$\mathcal{L}$	✓	X
Equation of Motion	E^A	✓	✓
Energy-momentum tensor	$T^{\mu\nu}$	✓	X
Angular momentum tensor	$M^{\lambda\mu\nu}$	✓	X

Figure 1.1: A comparison of the gauge invariance of different equations in electrodynamic and spin-2 theory. A ✓indicates gauge invariance, an X indicates gauge dependence. Complete gauge invariance requires a ✓for each quantity of the theory.

is particularly troubling in the context of Noether's first theorem, as we will see in the next section. The lack of a gauge invariant energy-momentum tensor in spin-2 causes problems with uniqueness, Noether's first theorem, gauge fixing and physical interpretation, to name a few. The convention of calling electrodynamics and spin-2 "gauge invariant" ignores unavoidable differences in the two theories; the convention to call a theory gauge invariant just because the equation of motion is exactly invariant is a weaker definition of gauge invariance than we consider in this thesis. For this reason, we refer to complete gauge invariance as in [14] to emphasize the stronger form of the definition of gauge invariance found in classical electrodynamic theory.

1.3 Noether's first theorem

1.3.1 Analytical mechanics

The result in the literature most central to this thesis is Noether's first theorem [159, 124, 87], and as we argue, the most fundamental result in mathematical physics. Most have heard of Noether's first theorem (commonly referred to as just Noether's theorem) as something which relates symmetries to conservation laws. In this section we will give the technical reason behind these general statements. To begin, we turn to the 1788 results of Lagrange in Mécanique Analytique [128]. Starting from the action functional (for example in electrodynamics),

$$S[A_\mu(x_\alpha)] = \int\int\int\int_{R_X} \mathcal{L} dx_1 dx_2 dx_3 dx_4, \tag{1.27}$$

one can derive the Euler-Lagrange equation of motion by using integration by parts to isolate the part proportional to the dependent variable δA_μ. Going forward we will abbreviate these integrals with $(\int_{R_X} = \int\cdots\int_{R_X})$, $(dX = dx_1 dx_2 \ldots dx_n)$, where R_x represents the boundary of the integrals (region of integration). The remaining piece (under a total derivative) is the so-called boundary term (converted to a boundary integral using Stokes' theorem),

$$\delta S = \int_{R_X} \left[\left(\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)} \right) \delta A_\nu + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)} \delta A_\nu \right) \right] dX. \tag{1.28}$$

The condition to minimize the action after neglecting the boundary term is simply the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)} = 0. \tag{1.29}$$

This equation of motion is just one equation that is part of a complete theory such as electrodynamics, as discussed in the previous section. Therefore, from the Lagrange perspective the analytical approach is insufficient in deriving equations such as the conservation laws of the theory. This gap is where Noether's first theorem comes in so powerfully. To some degree the boundary term in the variation of the action we so readily discard is what she showed plays the role of conservation laws of the theory; but it is not so trivial.

1.3.2 Noether's first theorem

Noether placed a very specific condition on the difference in the action in transformed dependent and independent variables, such that the action remains invariant under the simultaneous variations of the independent variables (e.g. coordinates) and dependent variables (e.g. fields) of the action,

$$\Delta S = S[A^*_\mu(x^*_\alpha)] - S[A_\mu(x_\alpha)]. \tag{1.30}$$

The basic idea is that for the variation we will consider the linear part of this change to determine δS by Taylor expanding each of the primed variables about the non-primed variables and keeping linear in ϵ. Expanding each and keeping linear terms we are left with the variation [87],

$$\begin{aligned}\delta S = \int_{R_X} [\frac{\partial \mathcal{L}}{\partial x_\beta}\delta x_\beta + \frac{\partial \mathcal{L}}{\partial A_\nu}\partial^\beta A_\nu \delta x_\beta + \frac{\partial \mathcal{L}}{\partial A_\nu}\delta \bar{A}_\nu \\ + \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)}\partial_\rho \delta \bar{A}_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)}[\partial_\rho \partial^\beta A_\nu]\delta x_\beta + \mathcal{L}\partial^\beta \delta x_\beta]dX,\end{aligned} \tag{1.31}$$

where the bar variations have an essential distinction from the non-bar variations. The bar variations are the difference in transformed and non-transformed fields in the non-transformed coordinates $\delta \bar{A}_\nu = A^*_\nu(x_\alpha) - A_\nu(x_\alpha)$. The non-bar variations are the difference in transformed fields in transformed coordinates with the non-transformed fields in the non-transformed coordiantes $\delta A_\nu = A^*_\nu(x^*_\alpha) - A_\nu(x_\alpha)$. These two definitions are related by,

$$\delta \bar{A}_\nu = \delta A_\nu - \partial^\beta A_\nu \delta x_\beta. \tag{1.32}$$

From the variation (1.31) we proceed as in the case of the Lagrange approach, combining terms proportional to δA_μ and the rest under a total divergence. Only this time we keep the total divergence yielding,

$$\delta S = \int_{R_X} [\left(\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)}\right)\delta \bar{A}_\nu + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)}\delta \bar{A}_\nu + \eta^{\rho\beta}\mathcal{L}\delta x_\beta\right)]dX. \tag{1.33}$$

This is the identity derived from Noether's first theorem in the case of electrodynamics. It is an identity that relates the Euler-Lagrange equation of motion to the conservation laws of the theory.

We note that this derivation is fairly non-trivial and highly recommend following the book of Gelfand and Fomin [87] and Noether's paper [159] if one wishes to derive it for themselves.

In the physics literature, famous books such as Peskin and Shroeder [165] in their equation 2.11, present short-cut approaches based on obtaining a known result that do not follow the proper mathematical derivation found in e.g. Noether [159] or Gelfand and Fomin [87]. This approach is common in many physics textbooks [114, 197, 182] and articles [33, 46, 111, 161] that wish to apply Noether's first theorem. These approaches fail to obtain the crucial Equation (1.32) which is required for proper application of Noether's first theorem, and misses the final term in (1.33). They then have a limited set of transformations that lead to the canonical Noether energy-momentum tensor, which is not the physical energy-momentum tensor for most theories. In essence these methods rely on the Lagrange style action condition [165] without following the condition imposed by Noether (1.30), which is why terms are missing compared to the Noether approach. These approaches have a long history which likely stem from a late translation of Noether's paper into English [160]. Another potential root of this problem has been asserted the use of this so-called "Noether" result is done without ever actually reading her paper, as noted by Kastrup in 1984 [116]:

> "*I suspect -perhaps unfairly so- that even in recent years only a few of those authors who quote Noether's work or refer to her theorem had a chance to see or study the original publication.*"

Lanczos put this more bluntly in his article "Emmy Noether and calculus of variations" as [130]:

> "*Every theoretical physicist is familiar with the expression "Noether's theorem" or "Noether's principle," although none of them actually reads Emmy Noether's original paper.*"

In the words of leading Noether historian and mathematican Yvette Kosmann-Schwarzbach [125] "*Therefore, caveat lector! It is better to read the original than to rely on second-hand accounts.*". This issue — with "proper" vs. "improper" derivations of Noether's first theorem, is not one which we explore in this thesis. We note however that throughout Kosmann-Schwarzbach's work she continually makes note of the past omission of Noether in the physics community, statements such as "*These remarks in fact suggest that, even though the subject of Noether's article had been central to Wigner's preoccupations since the 1920s, he had never read the original paper*" [124] can be found throughout her book.

Chapter 2 is heavily dedicated to a related problem however, the improvement of the canonical Noether energy-momentum tensor to obtain desired physical results (See Section 1.5.1 for our discussion of this problem) — however it may very well be that this problem originates from incomplete derivations of Noether's first theorem found throughout the physics literature. We note that there are physics texts, such as [39, 183, 137], that have more of a proper Noether's theorem and variational symmetry derivation — however such presentation is less common in the literature. No source is as clear or detailed in this derivation as Gelfand and Fomin [87].

1.3.3 Noether's first theorem and electrodynamics

In the case of electrodynamics, we can obtain the equations of motion, all 15 conservation laws, and the Lorentz force from this identity. In other words, the complete set of equations for the theory are obtained from one compact identity,

$$\delta S = \int_{R_X} [\left(\partial_\rho F^{\rho\nu}\right)\delta\bar{A}_\nu + \partial_\rho\left(T^{\rho\beta}a_\beta + M^{\rho\alpha\beta}\omega_{\alpha\beta} + C^{\rho\alpha}\xi_\alpha + D^\rho S\right)]dX, \tag{1.34}$$

where the parameters $a_\beta, \omega_{\alpha\beta}, \xi_\alpha, S$ are the 15 parameters associated to the conformal group, presented in (2.20). The conserved tensors are exactly those presented in Section 1.1.2. These fundamental equations to the theory once seemingly disconnected results that must be pieced together to complete the set of equations of the theory are elegantly intertwined by Noether's first theorem. This gives the possibility for a concrete procedure for deriving the complete set of equations of a theory provided the Lagrangian density is known; it is hard to imagine a more powerful statement in mathematical physics.

1.3.4 The Bessel-Hagen method

A contemporary of Noether, Erich Bessel-Hagen, solved this problem in 1921 by considering the complete set of variational symmetries of electrodynamics (coordinate and gauge) to derive all 15 physical conservation laws [26]. This paper was known in the mathematics literature for other reasons (non-exact symmetries and Noether's theorem), but has been almost complete ignored in the physics literature. This led to physicists believing that the canonical Noether expression must be ad-hoc improved using, for example, the superpotential method, and Belinfante procedure [23]. One possible explanation is that proper application using the Bessel-Hagen method was not translated into English until 2006 [108]. Proper application of Noether's first theorem, its converse, the superpotentials and improvement problem, and the Bessel-Hagen method applied to several examples are some of the major topics of Chapter 2, so we will not go into more details here.

We will now briefly summarize the Bessel-Hagen application of Noether's first theorem to electrodynamics. Bessel-Hagen showed by considering both coordinate and gauge symmetries of the action, the proper variational symmetry of the fields $\delta\bar{A}_\mu$ is,

$$\delta\bar{A}_\alpha = -F^\nu{}_\alpha \delta x_\nu. \tag{1.35}$$

Substituting this $\delta\bar{A}_\alpha$ into (1.33) we obtain the following identity,

$$\delta S = \int_{R_X} [\left(\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\right)\delta\bar{A}_\nu + \partial_\rho \left(T^{\rho\beta}\delta x_\beta\right)]dX. \tag{1.36}$$

We now require coordinate symmetries of the action δx_μ. In 1909-1910, Bateman [17, 18] determined that electrodynamics was invariant under the 15 parameter conformal group. These 15 parameters include the 10 parameter Poincare group (6 of which form the Lorentz group) associated to the energy-momentum and angular momentum in the Noether picture. The 5 remaining parameters are associated to the conformal and dilatation tensors.

In [174] a self-contained derivation of the 15 parameter conformal symmetries of electrodynamics is presented. More commonly the conformal Killing's equation is solved for the Minkowski metric isometries to determine these δx_μ as,

$$\delta x_\alpha = a_\alpha + \omega_{\alpha\beta}x^\beta + S\, x_\alpha + 2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu, \tag{1.37}$$

where a_μ is associated to the 4 parameter Poincare translation, and $\omega_{\alpha\beta}$ is associated to the 6 parameter Lorentz group. The remaining 5 parameters S and ξ_ν are associated to the remainder of the 15 parameter conformal group. Substituting these δx_μ into (1.36) yields the complete set, which is exactly equation (1.34) for electrodynamics if we factor out the parameters in the conformal transformations. We have all 15 conservation laws of electrodynamics directly derived from Noether's first theorem. From $\partial_\mu T^{\mu\nu}$ we can use on-shell conditions to obtain the Lorentz force law. Therefore, Noether's first theorem gives us a concrete procedure for deriving the complete set of equations of electrodynamic theory given only the Lagrangian density of the theory.

1.3.5 The canonical Noether energy-momentum tensor

For these reasons Noether's first theorem receives great praise in the literature and scientific community as "*one of the most amazing and useful theorems in physics*" [101]. However, closer examination shows that conventional presentation of the "canonical Noether energy-momentum tensor" $T_C^{\rho\beta}$ is not working to derive the physical conservation laws based on the physical $T^{\rho\beta}$ in most cases such as electrodynamics [76]. In the case of electrodynamics that canonical Noether tensor is,

$$T_C^{\rho\beta} = -\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\nu)}\partial^\beta A_\nu + \eta^{\rho\beta}\mathcal{L} = F^{\rho\nu}\partial^\beta A_\nu - \frac{1}{4}\eta^{\rho\beta}F_{\mu\nu}F^{\mu\nu}, \tag{1.38}$$

which is not symmetric, not gauge invariant, and not tracefree. Although we note that the canonical "Noether" expression has nothing to do with Noether's results, this was named after her due to the ability to derive it from her theorem. The origin of this problem is applying only the 4-parameter Poincare translation to Noether's first theorem instead of the complete set of variational symmetries of the action. We discuss this canonical tensor at length in Chapter 2.

1.3.6 Noether's first theorem overview

For the reasons given in the previous section, given Noether's first theorem and the Lagrangian density, all subsequent equations of motion and conservation laws are implicit information to the Lagrangian density:

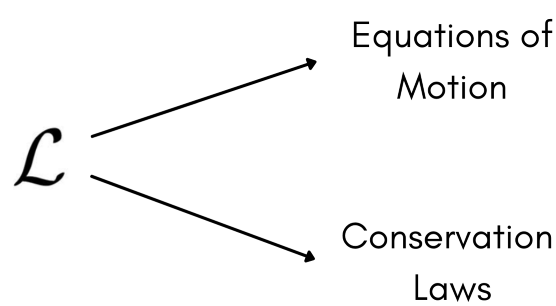

Figure 1.2: A simple schematic describing the application of Noether's theorem to derive both equations of motion and conservation laws from a single Lagrangian density.

It is for this reason that the Lagrangian density (action) represents the full set of equations and that Noether's first theorem is the key to unlocking this set. For example in the case of electrodynamics we have:

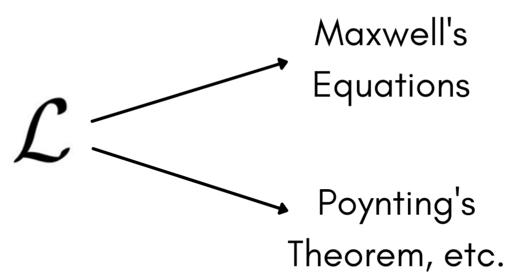

Figure 1.3: A simple schematic describing the application of Noether's theorem to derive both equations of motion and conservation laws from a single Lagrangian density in the case of electrodynamic theory.

The motivations, goals, and outcomes of this thesis center around Noether's first theorem. It is only by use of Noether's first theorem that we can define the axiomatic approaches we take in this thesis.

1.4 An Axiomatic Approach to Field Theory

Our thesis is guided by our goal to realize an axiomatic approach to physics which we will define in this section. Before doing so we wish to be specific about what is meant by an axiomatic approach to physics, because the term ”axiom” takes on different meanings in different disciplines, so we do not wish to conflate them. The axiomatic approach to physics is loosely defined as writing down a set of rules (physical requirements) that uniquely specify some physical law(s) — following some procedure defined by these axioms one can recover some physical theory or theories. We take this definition and when we refer to “axiom” we synonymously are referring to “physical requirements”.

To a mathematician/ logicist, an axiom is a mathematical statement (relationship) that is absolutely true within a system of logic. Even for this strict definition there is debate as to whether or not such an axiom has any fundamentally significant meaning [138]. Therefore physics, an experimental science, cannot hold such a strict definition of an axiom as would be considered in the mathematics community. To these communities, what we are proposing would be postulates or rules (physical requirements) based on a large body of evidence rather than some required absolute truth. However, the nature of our approach (whether misnamed or otherwise) is known within the physics community as the axiomatic approach to physics, therefore we will continue with our terminology regardless of any objections that may exist outside of the field.

1.4.1 Past axiomatic approaches to physics

In the 1950s-1960s the terminology axiomatic approach to physics was popularized. The popularizing of this terminology came through quantum field theory where the axiomatic approach to quantum field theory, initiated by Wightman and the Garding-Wightman axioms [193]. Furthermore, the Haag-Kastler axioms [99] amongst others continued this approach of axiomatically formulating quantum field theory. A major motivation of these works was different than ours, they were motivating by isolating a set of minimum axioms to help with understanding some consistency issues in quantum field theory. Groups such as Gergely Székely's (see [185]) have considered axiomatization of relativity theory. The works on axiomatization of theories, as we will discuss, are completed using quite different approaches than we take in this thesis. However, the premise taken by these pioneers is similar to that which we are taking. First, important desired properties of quantum field theories are identified and given the status of a fundamental axiom. Second, these axioms must be translated into recovering the mathematical model describing the physical phenomena.

Authors have recently taken the axiomatic approach to physics to areas such as quantum mechanics [143] where axioms (physical requirements) are listed which uniquely specify quantum mechanics as a single model out of a universal set of all possible probabilistic models. This approach is similar, but has a key distinction, to our own. Indeed, we consider the universal set of all possible classical field theories within our domain and hope to recover physical model(s) as the intersection of the subsets of the axiomatic conditions. While in [143] the goal is indeed to determine the model at the intersection of the subsets, their methods recover just a single physical model. This is similar to the axiomatic approaches to quantum field theory that prescribe axioms to recover a single model:

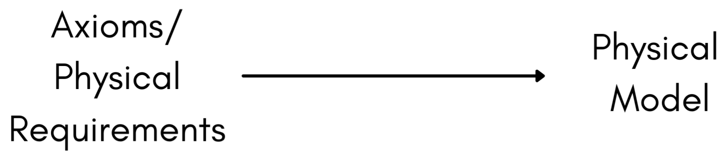

Figure 1.4: Conventional axiomatic approaches in physics, defining a set of physical requirements such that one unique physical model (theory) can be obtained from these axioms.

In field theories such as electrodynamics similar one-to-one considerations can be found on the arXiv [98]. The primary criticism with these approaches is that they have a one to one correspondence between a set of axioms and a set of physical equations. Therefore it is not clear whether the axioms provide any additional value in such cases, or if the content of the theory has been re-expressed in other terms. Basically this is a ”circular” criticism of axiomatic approaches that have one-to-one correspondence between axioms and the theory:

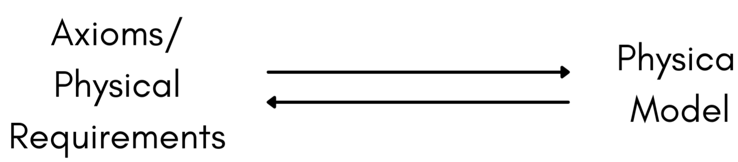

Figure 1.5: Criticism that one-to-one axiomatic approaches are circular (axioms just re-write the content of the theory in other terms).

In this thesis we wish to avoid such possible criticisms of our axiomatic approach by strictly requiring one-to-many correspondence from the axioms to a set of physical theories which follow from a common set of axioms. However, we note that we do not personally criticize such approaches in the past: formulating a set of axioms corresponding to specific field theories can been very insightful into making physical requirements of a theory more apparent and even lead into new insight not realized by the mathematical description of the theory [58]. Regardless, we take an altogether different approach than the standard literature and the goals, problems addressed and results of this thesis for the most part differ from the scope of the standard literature on axiomatic approaches to physics.

1.4.2 An axiomatic approach to field theory

We now turn to our axiomatic approach to field theory. Our goal is to consider axioms (some procedure based on physical requirements which will constrain possible physical models) which can be used to obtain numerous models from a common set of axioms:

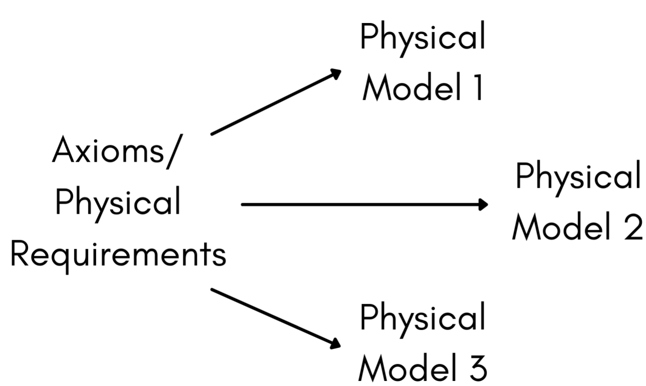

Figure 1.6: One of the proposed goals of the thesis, to define a procedure which yields multiple distinct physical models from a common set of axioms.

Given electrodynamics and Noether's first theorem we have an idealistic view of what a physical field theory can be. If we have the set of Lagrangian densities for our fundamental field theories, we can apply Noether's first theorem to derive the complete set of equations for each theory:

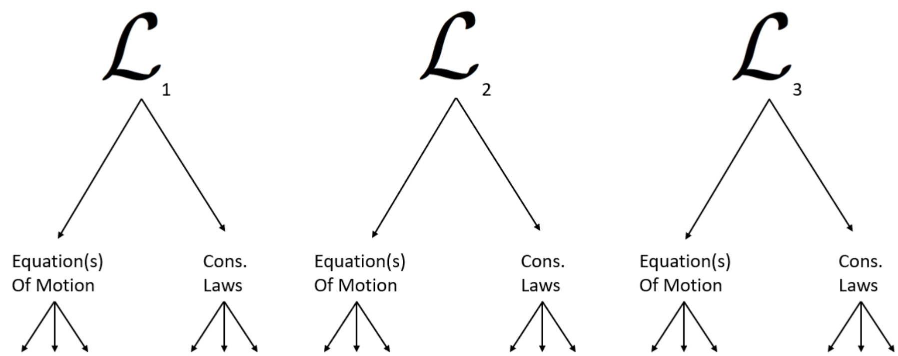

Figure 1.7: The complete set of physical equations can be considered implicit information to the Lagrangian density if Noether's first theorem is concretely working.

This idealistic view suggests that the complete sets of equations are implicit information to the set of Lagrangian densities. What we wish to push further is the possibility to write down some set of axioms such that the set of Lagrangian densities can be obtained from a procedure based on a common set of axioms:

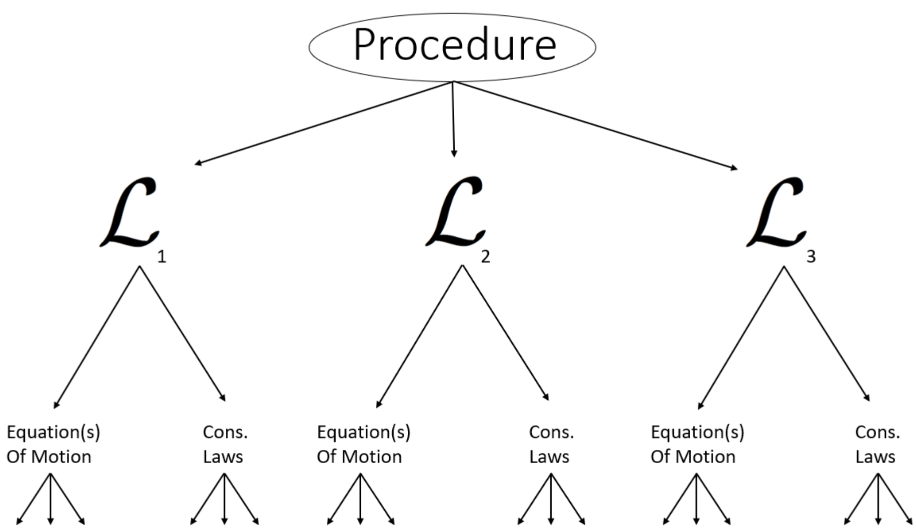

Figure 1.8: The idea is that some procedure, based on a common set of axioms, will yield multiple different Lagrangian densities which correspond to known physical models. This would render the set of Lagrangian densities implicit information to the set of axioms.

This idealistic view would render the set of Lagrangian densities implicit information to the set of axioms, allowing for the axioms to contain all necessary information for obtaining the complete sets of equations for the theories associated to the fundamental Lagrangian densities.

In some sense our idealistic view is in line with Dirac's view on the future of theoretical physics [64], which we share in its entirety in the epigraph of this thesis:

> "*The theoretical worker in the future will therefore have to proceed in an indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formulation that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities (by a process like Eddington's Principle of Identification).*"

We share this view of Dirac, and only differ on the point that in this thesis we do not propose new mathematical features in terms of physical entities, rather we wish recover existing well established physical models from our procedure. There is the possibility however, that our approach can yield unforeseen models, that may some day be linked to a physical theory, completing the view of Dirac. One of the main goals of this thesis is to see how far our idealistic view can be realized, with its implementation of obtaining Lagrangian densities from procedure based on physical requirements being the focus of Chapter 4 [14, 9, 11].

1.5 Problems preventing implementation of our axiomatic approach

This idealistic view has some problems which, at present day, prevent it from being applicable in all branches of physics. For example, general relativity is invariant under an infinite group of transformations which make application of Noether's first theorem problematic; the finite set of global conservation laws in the case of electrodynamics are due to the universality of Minkowski spacetime. For this reason, our focus in this thesis is on special relativistic field theory. As we will see, even for special relativistic field theory there are several roadblocks preventing the realization of the ideal view. Throughout all 3 chapters of the thesis, we provide solutions to these existing problems. The problems and motivation to solve them will form the remainder of this introduction. Note that application of general relativity to Noether's second theorem is possible (infinite groups of transformations), and using this identity one can derive diffeomorphism invariance of the theory [119]. General relativity is the topic of the final section of Noether's paper, however most of our discussion of GR will be through the linearized version of the theory (spin-2) which is a special relativistic field theory.

Noether's second theorem, while not the focus of this thesis, can be applied in the special relativistic gauge theories we consider to derive the gauge symmetries we require in the

Bessel-Hagen approach. For this reason, if one hopes to derive the coordinate and field (gauge) symmetries of a given theory, Killing's equation and Noether's second theorem can be used for systematically obtaining each, respectively. We will now turn to problems with the proposed axiomatic approach in the literature which we propose solutions to in this thesis. Each of the three body chapters of the thesis correspond to the three following problems, respectively.

1.5.1 Problem 1: Noether's first theorem and the improvement problem

The first problem is that the aforementioned canonical Noether energy-momentum tensor derived from Noether's first theorem using the 4-parameter Poincare translation does not correspond to the accepted physical energy-momentum tensor in most standard field theories [76]. The physical energy-momentum tensor, as defined by Blashcke et al. [29] is the unique, on-shell conserved, symmetries, gauge invariant and trace-free energy-momentum tensor associated to a particular field theory. This definition is motivated by theories such as electrodynamics and Yang-Mills theory who energy-momentum tensor simultaneously has all of these properties.

Since the canonical Noether tensor does not correspond to this physical energy-momentum tensor, the impression is given that one of physics most celebrated results, Noether's first theorem, is in some sense not working. To fix this problem, various ad-hoc "improvements" have been proposed throughout the years that add terms to the canonical expression out of nowhere to obtain the desired physical expression. Convention is to name these ad-hoc additions "improvements", since they "improve" the canonical Noether energy-momentum tensor. However, the improvements have nothing to do with Noether's theorems, and even the canonical tensor itself was not introduced by Noether. As noted by Forger and Romer in a leading review on energy-momentum tensors [76]:

> "*There is a long history of attempts to cure these diseases and arrive at the physically correct energy-momentum tensor $T_{\mu\nu}$ by adding judiciously chosen "improvement" terms to $\Theta^{\mu\nu}$ [$T_C^{\mu\nu}$ in our notation] (p. 307),*"

they then go on to discuss the flaw of the improvement methods,

> "*However, all these methods of defining improved energy-momentum tensors are largely "ad hoc" prescriptions focused on special models of field theory, often geared to the needs of quantum field theory and ungeometric in spirit.*"

Similar statements regarding the frustration this places on those who wish to study Noether's theorem was stated by Munoz [155],

> "*Few things are more frustrating to students than to be led through a long, formal argument only to be told at the end that the result obtained is incorrect and must somehow be fixed by an auxiliary procedure. This is particularly harmful if the formal argument involved turns out to be one of the mathematical cornerstones of modern physics ... the students will be left with the paradoxical feeling that a supposedly very general theorem produces unacceptable answers when applied to certain specific situations,*"

he also discusses the common impression this leaves on one who studies Noether's first theorem applied to field theories,

> "*they walk away with the impression that Noether's theorem somehow fails when spacetime symmetries are involved.*"

The solution to, and history of this problem is the focus of Chapter 2 so we will not go into more detail here: the Bessel-Hagen methodology fixes the issue for all completely gauge invariant special relativistic field theories we considered. For our purpose with the axiomatic approach, and more broadly gauge theory in general, this is an essential step in realizing the idealistic goals. For the complete set of equations in a given theory to be considered implicit information to the Lagrangian density, the procedure which realizes this information (Noether's first theorem) must be able to obtain the complete set of equations without any ad-hoc manipulations required. The Bessel-Hagen method outlined in Chapter 2 provides exactly this procedure; Noether's first theorem is working just fine to derive complete sets of equations without the need for improvements. We extend this method to several theories beyond electrodynamics: Yang-Mills theory, linearized Gauss-Bonnet gravity, Kalb-Ramond and totally antisymmetric fields. Only the Yang-Mills variational symmetry was "known",

$$\bar{\delta}A^a_\mu = -F^a_{\mu\nu}\delta x^\nu, \tag{1.39}$$

but it was conjected by Jackiw in [110], without deriving it or connecting it to Bessel-Hagen in any way. The variational symmetry associated to linearized Gauss-Bonnet gravity,

$$\delta\bar{h}_{\rho\sigma} = -2\Gamma^\nu{}_{\rho\sigma}\delta x_\nu, \tag{1.40}$$

is one of our main results [14], showing that for a gravitational model the field symmetries are proportional to the Christoffel symbol, analogous to the proportional to field strength results in the other theories we consider.

Our results in Chapter 2 are strong but are only for completely gauge invariant theories. In the case of theories with only a gauge invariant equation of motion such as spin-2, we have not yet applied these methods, however this application is the subject of future work. Spin-2 is discussed through the thesis and is the focus of [8]. We did not apply it in this thesis because there are several existing barriers to its application. First it requires the non-exact symmetry method of Bessel-Hagen, which is more complicated than the exact symmetries we considered. The main barrier is the non-uniqueness of the energy-momentum tensor; several expressions exist in the published literature [28], none of which are gauge invariant [140], therefore it is not clear which should be used to write down conservation laws for the theory.

The spin-2 non-uniqueness problem is central to the Padmanabhan-Deser debate [163], where a unique spin-2 energy-momentum tensor is required to complete a supposed derivation of general relativity from spin-2 theory. Padmanabhan showed that the conventional choice does not in fact work to derive Einstein's field equations or the Einstein-Hilbert action; only yet another expression can possibly obtain Einstein's field equations, and not the Einstein-Hilbert action. Therefore, conclusions of the fundamental spin-2 to GR results depend on which $T^{\mu\nu}$ is selected, and perhaps none of them work to recover the full set of Einstein's field equations and the Einstein-Hilbert action.

Others argue that energy-momentum tensors in spin-2 are not physical to begin with [65], but we will not get into this philosophical debate at this time. Regardless, calculations requiring a unique $T^{\mu\nu}$, and conservation laws of the theory, simply have no sense until a unique $T^{\mu\nu}$ can be determined and agreed upon. See [8] in Chapter 3 for our extensive contribution in this direction.

1.5.2 Problem 2: Multiple definitions of the energy-momentum tensor

Another main problem with realizing the idealistic view is that multiple methods for deriving energy-momentum tensors exist in the literature [75, 29, 23, 45]. Most notably, they include the canonical Noether method, the Hilbert (metric) method, the Fock method [75], the Bessel-Hagen (proper) Noether method [26] discussed in Chapter 2, and various improvement methods such as [23, 45]. This practice is contrary to deriving the equation of motion for a particular theory, where the Euler-Lagrange equation is uniquely used as the procedure to do so. The focus of Chapter 3 is to clarify the relationship amongst these different definitions and move towards a unique definition that can end ambiguity problems and provide a concrete procedure for realizing the ideal goals.

Significant work in this direction has been published regarding the relationship between two of these definitions: the Hilbert (metric) and canonical-Noether energy-momentum tensors [176, 133, 95, 170]. These articles in various ways come to the same conclusion, that the Hilbert (metric) tensor, and the Belinfante improved canonical-Noether tensor are proportional to each other (and equivalent on-shell); [170] gives a proof in the case of first order vector field theories. In [8] we prove that this association is off-shell not unique in the spin-2 case, in other words, that there are infinitely many off-shell relations, only one of which is the Hilbert tensor. The Hilbert (metric) energy-momentum tensor is familiarly known as the source of Einstein's field equations. This definition, involves variations of a curved space Lagrangian with respect to the metric tensor,

$$T_{\gamma\rho} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\gamma\rho}}. \tag{1.41}$$

A slight variation of this tensor is used to derive the energy-momentum tensor for a special relativistic field theory, known as the Hilbert (metric) energy-momentum tensor in Minkowski spacetime [29]. The Hilbert energy-momentum tensor in Minkowski spacetime is derived by writing the action in 'curved space' by replacing all ordinary derivatives with covariant derivatives $\partial \to \nabla$, replacing all Minkowski metrics with general metrics $\eta \to g$, and introducing the Jacobian term $\sqrt{-g}$. The curved space Lagrangian is then varied with respect to the metric tensor and "returned" to Minkowski spacetime according to,

$$T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\gamma\rho}}\bigg|_{g=\eta}. \tag{1.42}$$

In many trivial cases such as electrodynamics this yields the unique physical energy-momentum tensor of the theory [29]. This has been one of the root causes for the conventional wisdom that the various methods are in some sense generally equivalent. One of our main contributions in this thesis is a formal disproof of this notion in [13] "Noether and Hilbert (metric) energy-momentum tensors are not, in general, equivalent", published in Nuclear Physics B. In this article we prove that the Noether energy-momentum tensor derived directly from the Bessel-Hagen method (the physical energy-momentum tensor) is not always the same as the Hilbert (metric) energy-momentum tensor in Minkowski spacetime,

$$T_H^{\gamma\rho} \neq T_N^{\gamma\rho}. \tag{1.43}$$

The counterexample we used for this disproof is the linearized Gauss-Bonnet gravity model [14, 9], because it has a well-established unique energy-momentum tensor [156], and has an action with second order derivatives of a second rank tensor potential. The problem with past claims of equivalence were that only simple actions with first order derivatives of first rank tensor (vector) potentials were considered. In more nontrivial cases we show in [13] that the covariant derivatives of higher rank potentials create additional terms which diverge the result from the Noether method. See [13] for our extensive discussion on this topic.

The Noether/ improvement and Hilbert methods are, as noted by Blashcke et al., "*conceptually and mathematically quite different*", so while our result questions the conventional wisdom, it is not really a surprise. The published literature in this direction [176, 133, 95, 170] focused on the question of relating the canonical Noether and Hilbert energy-momentum tensors, whereas in [13] we focus on the physical energy-momentum tensor obtained directly from Noether's first theorem without the need for any ad-hoc improvement terms. The general idea of the studies [176, 133, 95, 170] is to compare the canonical Noether, plus improvement terms, to the Hilbert tensor. The general conclusion agreed upon by these authors is that the Belinfante improvement (symmetrization) procedure is proportional to the Hilbert tensor in Minkowski spacetime. For example, in the case of electrodynamics the relationship to the Hilbert tensor is well established,

$$T_H^{\mu\nu} = T_C^{\mu\nu} + \partial_\alpha(F^{\alpha\mu}A^\nu) - A^\nu E^\mu, \tag{1.44}$$

where the superpotential and on-shell terms are the "improvements" that relate the Hilbert expression to the canonical Noether expression. The first two terms are the Belinfante result $T_B^{\mu\nu} = T_C^{\mu\nu} + \partial_\alpha(F^{\alpha\mu}A^\nu)$, where $b^{[\alpha\mu]\nu} = F^{\alpha\mu}A^\nu$ is the Belinfante superpotential. We have independently confirmed these results for a vector field as in [170] (not included in the thesis) and for spin-2 Fierz-Pauli theory in [8],

$$T_H^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\gamma \Psi_b^{[\rho\gamma]\sigma} - 2h_\beta^\sigma E^{\rho\beta}, \tag{1.45}$$

where the Belinfante superpotential for spin-2 Fierz-Pauli theory is,

$$\begin{aligned}\Psi_b^{[\rho\gamma]\sigma} = \frac{1}{2}\eta^{\rho\sigma}h^{\gamma\alpha}\partial_\alpha h_\beta^\beta - \frac{1}{2}\eta^{\gamma\sigma}h^{\rho\alpha}\partial_\alpha h_\beta^\beta + \frac{1}{2}h^{\gamma\sigma}\partial^\rho h_\beta^\beta - \frac{1}{2}h^{\rho\sigma}\partial^\gamma h_\beta^\beta \\ + h^{\rho\lambda}\partial_\lambda h^{\gamma\sigma} - h^{\gamma\lambda}\partial_\lambda h^{\rho\sigma} + h^{\sigma\beta}\partial^\gamma h_\beta^\rho - h^{\sigma\beta}\partial^\rho h_\beta^\gamma.\end{aligned} \tag{1.46}$$

See [15] for more discussion of the Belinfante method. The Hilbert and Belinfante tensors are merely proportional to one another, differing by terms proportional to the equations of motion in addition to the superpotential term. For this reason, the Belinfante improved tensor is on-shell equivalent to Hilbert (removing terms proportional to the equations of motion). This is a much weaker and less desirable trait than proper application of the Bessel-Hagen method, where the physical energy-momentum tensor is directly obtained off-shell with no improvements needed; there are numerous reasons for the superiority of this method, as discussed throughout Chapter 2.

The questions still remains as to the status of the many non Bessel-Hagen methods (i.e.

canonical Noether, Hilbert, Fock, Belinfante, and other improvement methods). In simple examples they coincide, but is this by coincidence? Or does some meaningful relationship exist? We explore this question for the simplest model, the Klein-Gordon scalar field, in [10]. Instead of considering the trivial Lagrangian that allows these various methods to coincide, we consider the general system of Lagrangian which yield the Klein-Gordon equation of motion, with free coefficients on each term. Solving for the coefficients we prove several results about the relationship between the different definitions for $T^{\mu\nu}$, such as the divergence of results of the methods for certain solutions, and the existence of an off-shell traceless expression.

This existence of an off-shell traceless expression is notably stronger than the on-shell traceless "new improved" energy-momentum tensor [45] which "improves" the canonical Noether tensor to an on-shell traceless expression (traceless being a necessary property in e.g. conformal field theory). What we emphasize form this contribution is that the many different "energy-momentum" tensor definitions are motivated by them coinciding in trivial examples, not by general proofs of equivalence, and that even for scalar fields the relationships can be broken down. Calling numerous mathematically distinct procedures by the same symbol "$T^{\mu\nu}$" and name "energy-momentum tensor" we argue is a problematic practice and should be replaced by a unique and clear methodology; we argue in favour of the Bessel-Hagen approach to Noether's first theorem. Our final contribution to Chapter 3 is the study on the canonical Noether energy-momentum tensor for spin-2 Fierz-Pauli theory. Due to the conventional wisdom that one can add improvements to the canonical tensor in the form of the divergence of a superpotential and on-shell terms, all of the non-unique spin-2 energy-momentum tensors can be recovered from improving the canonical Noether expression. In the case of the Landau-Lifshitz tensor for example,

$$T^{\mu\nu}_{LL} = T^{\mu\nu}_{C} + \partial_\alpha \Psi^{[\mu\alpha]\nu}_{LL} + hE^{\mu\nu} - 2h^{\nu}_{\beta}E^{\mu\beta}, \tag{1.47}$$

where the superpotential required for this model is,

$$\begin{aligned}\Psi^{[\mu\alpha]\nu}_{LL} = \frac{1}{2}[&\eta^{\mu\nu}h\partial^\alpha h - \eta^{\nu\alpha}h\partial^\mu h + \eta^{\nu\alpha}h\partial_\beta h^{\mu\beta} - \eta^{\mu\nu}h\partial_\beta h^{\alpha\beta} + h\partial^\mu h^{\nu\alpha} - h\partial^\alpha h^{\mu\nu}] \\ &+ h^{\nu\alpha}\partial^\mu h - h^{\mu\nu}\partial^\alpha h + h^{\mu\nu}\partial_\beta h^{\alpha\beta} - h^{\nu\alpha}\partial_\lambda h^{\mu\lambda} + h^{\nu}_{\beta}\partial^\alpha h^{\mu\beta} - h^{\nu}_{\beta}\partial^\mu h^{\beta\alpha}.\end{aligned} \tag{1.48}$$

These relationships give the impression that all of these expressions have some meaningful connection to the Noether methodology (although recall the "canonical Noether energy-momentum tensor" was not one of Noether's results, despite being derived from her theorem).

The problem with allowing these ad-hoc "improvements" is that for the ambiguity problems, such as the aforementioned Padmanabhan-Deser debate [163], no logical progress can be made in the energy-momentum laws of the theory — if everything can be ad-hoc "improved" to come from the same canonical Noether expression, then contradictory calculations and conservation laws are also justified. Of the 10-20 published expressions for spin-2 [8], all are conserved on-shell, and obtainable by improving the canonical $T_C^{\mu\nu}$. To explore the general case for spin-2 theory, in [8] we write down the most general (Fock) energy-momentum tensor,

$$
\begin{aligned}
T^{\rho\sigma} = {}& b_1\partial_\alpha h^{\rho\sigma}\partial_\beta h^{\alpha\beta} + b_2\partial_\alpha h^{\rho\sigma}\partial^\alpha h + b_3\partial_\alpha h^{\rho\alpha}\partial_\beta h^{\sigma\beta} + b_4\partial_\alpha h^{\rho}{}_{\beta}\partial^\alpha h^{\sigma\beta} + b_5\partial_\alpha h^{\rho\beta}\partial_\beta h^{\sigma\alpha} \\
&+ b_6\partial^\rho h\partial^\sigma h + b_7\partial^\rho h_{\alpha\beta}\partial^\sigma h^{\alpha\beta} + b_{8_i}\partial^\rho h^{\sigma\alpha}\partial_\alpha h + b_{8_{ii}}\partial^\sigma h^{\rho\alpha}\partial_\alpha h + b_{9_i}\partial^\rho h\partial_\alpha h^{\sigma\alpha} + b_{9_{ii}}\partial^\sigma h\partial_\alpha h^{\rho\alpha} \\
&+ b_{10_i}\partial^\rho h^{\sigma\alpha}\partial^\beta h_{\alpha\beta} + b_{10_{ii}}\partial^\sigma h^{\rho\alpha}\partial^\beta h_{\alpha\beta} + b_{11_i}\partial^\rho h_{\alpha\beta}\partial^\alpha h^{\sigma\beta} + b_{11_{ii}}\partial^\sigma h_{\alpha\beta}\partial^\alpha h^{\rho\beta} \\
&+ c_1\eta^{\rho\sigma}\partial_\alpha h\partial^\alpha h + c_2\eta^{\rho\sigma}\partial_\alpha h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + c_3\eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\lambda h^{\lambda}{}_{\beta} + c_4\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} + c_5\eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\beta h \\
&+ d_1 h^{\rho\sigma}\partial_\alpha\partial^\alpha h + d_2 h^{\rho\sigma}\partial_\alpha\partial_\beta h^{\alpha\beta} + d_3 h\partial_\alpha\partial^\alpha h^{\rho\sigma} + d_4 h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\rho\sigma} + d_{5_i}h^{\rho\alpha}\partial^\beta\partial_\beta h^{\sigma}{}_{\alpha} + d_{5_{ii}}h^{\sigma\alpha}\partial^\beta\partial_\beta h^{\rho}{}_{\alpha} \\
&+ d_{6_i}h^{\rho\alpha}\partial_\alpha\partial_\beta h^{\sigma\beta} + d_{6_{ii}}h^{\sigma\alpha}\partial_\alpha\partial_\beta h^{\rho\beta} + d_7 h\partial^\rho\partial^\sigma h + d_8 h_{\alpha\beta}\partial^\rho\partial^\sigma h^{\alpha\beta} \\
&+ d_{9_i}h^{\rho\alpha}\partial^\sigma\partial_\alpha h + d_{9_{ii}}h^{\sigma\alpha}\partial^\rho\partial_\alpha h + d_{10_i}h^{\rho\alpha}\partial^\sigma\partial^\beta h_{\alpha\beta} + d_{10_{ii}}h^{\sigma\alpha}\partial^\rho\partial^\beta h_{\alpha\beta} \\
&+ d_{11_i}h\partial^\rho\partial_\alpha h^{\sigma\alpha} + d_{11_{ii}}h\partial^\sigma\partial_\alpha h^{\rho\alpha} + d_{12_i}h_{\alpha\beta}\partial^\rho\partial^\alpha h^{\sigma\beta} + d_{12_{ii}}h_{\alpha\beta}\partial^\sigma\partial^\alpha h^{\rho\beta} \\
&+ a_1\eta^{\rho\sigma}h_{\alpha\beta}\partial^\alpha\partial^\beta h + a_2\eta^{\rho\sigma}h\partial_\alpha\partial_\beta h^{\alpha\beta} + a_3\eta^{\rho\sigma}h_{\alpha\beta}\partial^\alpha\partial_\lambda h^{\lambda\beta} + a_4\eta^{\rho\sigma}h\partial_\alpha\partial^\alpha h + a_5\eta^{\rho\sigma}h_{\alpha\beta}\partial_\lambda\partial^\lambda h^{\alpha\beta}.
\end{aligned}
\tag{1.49}
$$

We then write this most general system as the most general improved canonical Noether tensor plus improvement terms,

$$
T^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha\Psi^{[\rho\alpha]\sigma} + \zeta_1 hE^{\rho\sigma} + \zeta_2 h^{\rho}_{\alpha}E^{\sigma\alpha} + \zeta_3 h^{\rho\sigma}\mathbf{E} + \zeta_4 h^{\sigma}_{\alpha}E^{\rho\alpha} + \zeta_5\eta^{\rho\sigma}h\mathbf{E} + \zeta_6\eta^{\rho\sigma}h_{\alpha\beta}E^{\alpha\beta}. \tag{1.50}
$$

From here we solved the linear system of coefficients for a symmetric, conserved result. There are infinitely many solutions, of which the published expressions are all solutions to this system. In other words, the connection to the canonical Noether tensor from improvements is nothing special; we can do this for infinitely many symmetric, conserved $T^{\mu\nu}$ we write down. Just because we can add terms to a result from Noether's first theorem, does not mean the result should be in any way considered a result of Noether's methods. With respect to Noether's first theorem, these "improvements" are ad-hoc manipulations that band-aid the non-uniqueness problem and have prevented any meaningful progress in this direction. With no gauge invariant energy-momentum tensor [140], it is not clear how to solve the non-uniqueness problem.

Furthermore, it has been argued that none of these have been used for experimental or observation purposes [65], although we will stay away from this debate in this thesis.

Our main result in [8] is that there are infinitely many Belinfante improved tensors, and only one of which is the Hilbert Tensor. This is contrary to the conventional wisdom that this Belinfante improvement is uniquely associated to the Hilbert tensor. This result shows that even in the most supported improvement procedure, unique association to the canonical Noether tensor is not guaranteed; only through human selection of particular improvement terms can we recover the desired result. Extensive discussion of this result we leave to [8].

1.5.3 Problem 3: Deriving Lagrangian densities from a set of axioms

In Chapter 2 we address the first problem, which is required to obtain complete sets of equations from a Lagrangian density by following a concrete procedure. In Chapter 2 we address the non-uniqueness of methodology in energy-momentum derivation. But these problems fail to address the most important question: how can we convert a set of axioms into a set of Lagrangian densities which cover our fundamental theories?

This question is the focus of Chapter 4; in Chapter 4, we define a procedure from deriving Lagrangian densities from general linear combinations of scalars of N order of derivatives of an M rank tensor potential [14, 11]. For example, in the case of electrodynamics we have,

$$\mathcal{L} = a\partial_\mu A_\nu \partial^\mu A^\nu + b\partial_\mu A^\mu \partial_\nu A^\nu + c\partial_\mu A_\nu \partial^\nu A^\mu. \tag{1.51}$$

We can then impose axioms and solve for the resulting free coefficients such that the Lagrangian densities, and subsequent set of equations, satisfy the conditions defined by the axioms. In the case of $N = M = 1$, requiring complete gauge invariance under the gauge transformation (1.20), the unique solutions is the electrodynamic Lagrangian density with a free coefficient. The free coefficient is fixed in Noether's first theorem and electrodynamic theory is recovered from the axioms [14]. Note that in the case of electrodynamics the negative sign on the Lagrangian is fixed by demanding positive energy in the Hamiltonian; if the sign on the Lagrangian was positive there would be no lower bound on the energy of the system.

We then considered the most general scalar for $N = 1$, $M = 2$ as in the case of spin-2 Fierz-Pauli theory,

$$\mathcal{L} = A\partial_\mu h^\mu_\nu \partial^\nu h^\gamma_\gamma + B\partial_\mu h^\mu_\nu \partial_\gamma h^{\nu\gamma} + C\partial_\mu h^\nu_\nu \partial^\mu h^\gamma_\gamma + D\partial_\mu h_{\nu\gamma}\partial^\mu h^{\nu\gamma} + E\partial_\mu h_{\nu\gamma}\partial^\nu h^{\mu\gamma}, \tag{1.52}$$

and solving the system of free coefficients for a complete gauge invariant solution. Of course, there is no solution to this linear system based on proofs of no gauge invariant $T^{\mu\nu}$ and known non-exact gauge invariance of the Fierz-Pauli action. We verified these results and determined the general Fierz-Pauli Lagrangian which is invariant up to a boundary term. We then considered the general expression for an $N = M = 2$ Lagrangian, with higher order derivatives, and solved for invariance under the spin-2 gauge transformation. Remarkably, there was a solution, in the form of the contracted linearized Riemann tensors,

$$\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2, \tag{1.53}$$

where each R is the linearized form of the Riemannian tensor after $g_{\mu\nu} \to eta_{\mu\nu} + \lambda h_{\mu\nu}$. Solving for these free coefficients to have a complete gauge invariant model, there is a unique solution in the form of linearized Gauss-Bonnet gravity,

$$\mathcal{L} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \tag{1.54}$$

Perhaps the most significant result here was the aforementioned Noetherian variation $\delta\bar{h}_{\rho\sigma} = -2\Gamma^{\nu}{}_{\rho\sigma}\delta x_{\nu}$ proportional to the linearized Christoffel symbol. In short, a connection between electrodynamics and linearized Gauss-Bonnet gravity was established as the unique solution to the procedure for $N = M = 1$ and $N = M = 2$ for a common set of axioms; the first realization of our idealistic goals. Gauss-Bonnet gravity in the physics literature dates back to the result of Lanczos [129]. See [9] for a historical overview of the development of Gauss-Bonnet gravity.

In [9] we took this study one step further to the dual formulations of the theories and showed that both can be expressed in analogous dual formulations. In this formulation the linearized Gauss-Bonnet gravity model does not simply have no equation of motion (as is often implied by its characterization as a topological theory), rather its equation of motion is the Bianchi identity analogous to the homogeneous half of Maxwell's equations,

$$\partial_{\lambda}R_{\mu\nu\alpha\beta} + \partial_{\mu}R_{\nu\lambda\alpha\beta} + \partial_{\nu}R_{\lambda\mu\alpha\beta} = 0. \tag{1.55}$$

Generalizing the dual formulations to the curvature tensors of higher spin gauge theories yields the models known as the Maxwell-like higher spin gauge theories, which we discuss in both [9] and [11].

The third paper in this chapter [11] continues along this line by applying the procedure to the $N = M = n$ case. Solving for complete gauge invariance under the spin-n gauge transformations, one continues to yield contractions of curvature tensors at each n. This gives a clear procedure for deriving the curvature tensors of higher spin gauge theories that were previously postulated based on symmetry properties of the Riemann tensor. In all, Chapter 4 provides necessary steps towards realizing the ability to determine a complete set of Lagrangian densities from a common set of axioms. In our case this set is all contractions of the electrodynamics field strength tensor, linearized Riemannian tensors, and curvature tensors of higher spin gauge theories.

1.6 Thesis overview

The three problems we have introduced are addressed in the three body chapters of the thesis, respectively. In the second chapter we address the possibility of a concrete procedure for obtaining the complete set of equations for a field theory such that this set of equations is implicit information to the Lagrangian density. The Bessel-Hagen approach to Noether's first theorem provides exactly this procedure [15] and we detail application of this method for several gauge invariant Lagrangian densities [12]. In the third chapter we address the non-uniqueness and improvement of the energy-momentum tensor in the literature. We provide a proof that the Noether and Hilbert definitions are not, in general, equivalent [13], we prove several results for the relationship of the numerous distinct definitions that exist for $T^{\mu\nu}$ in the case of a scalar field [10], and show that there are infinitely many spin-2 energy-momentum tensors that can be connected to the canonical Noether tensor via ad-hoc improvements [8]. Finally, in the fourth chapter we develop a method for obtaining a set of Lagrangian densities form the common set of axioms [14, 9, 11]. Schematically these chapters address the following aspects of the ideal view:

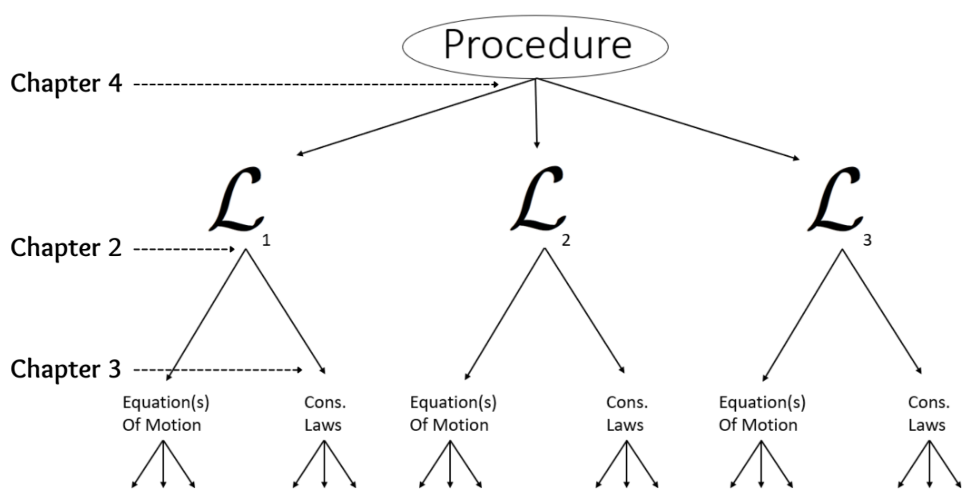

Figure 1.9: Each of the body chapters corresponds to a specific step of the axiomatic approach, corresponding to the 3 problems we have introduced, respectively.

At the start of each chapter we will briefly introduce the papers and topics included in the chapter. In the conclusions section we will summarize what we feel are the most significant results from the thesis, discuss their impact on existing results, discuss existing gaps in our methodology, and possible research that can stem from this thesis.

Chapter 2

Noether's first theorem and the Bessel-Hagen method

This chapter focuses on addressing the problem outlined in Section 1.5.1 of the Introduction: the need for concrete procedure for deriving complete sets of equations for a given physical theory in order to have an axiomatic approach to field theory. We show that the Bessel-Hagen approach to Noether's first theorem provides this concrete procedure for all complete gauge invariant field theories we consider. This chapter consists of two papers. The first paper [15] in Section 2.1 details problems with the conventional application of Noether's first theorem which does not obtain physical equations unless missing "improvement" terms are added ad-hoc, and we propose the converse of Noether's first theorem as a method for settling ambiguities in the various Noether methodology. This paper is to appear in the collected papers volume "*The Physics and Philosophy of Noether's Theorems*" published by Cambridge University Press. The second paper [12] in Section 2.2 focuses on the Bessel-Hagen approach to Noether's first theorem and successfully apply it to several models. This chapter provides concrete and clear methodology for obtaining complete sets of physical equations from a Lagrangian density, contrary to the conventional presentation of ad-hoc "improvement" (e.g. Belinfante) of the canonical Noether energy-momentum tensor. The papers in this chapter were each produced with two co-authors; for duties breakdown see the Co-Authorship Statement.

2.1 Converse of Noether's first theorem and the energy-momentum tensor ambiguity problem

Dedicated to the late Bessel-Hagen, who when alive had his habilitation thesis thrown into the sea, and even now must feel as if his work was lying somewhere on the seabed.

Abstract Noether's theorems are widely praised as being one of the most beautiful and useful results in all of physics. However, if one reads the majority of standard texts and literature on the application of Noether's first theorem to field theory, one immediately finds that the "canonical Noether energy-momentum tensor" derived from the 4-parameter translation of the Poincaré group does not correspond to what's widely accepted as the "physical" energy-momentum tensor for foundational theories such as electrodynamics. This gives the impression that Noether's first theorem is in some sense not working. In recognition of this issue, common practice is to "improve" the "canonical Noether energy-momentum tensor" by adding suitable ad-hoc "improvement" terms that will convert the canonical expression into the desired result. On the other hand, a less common but distinct method developed by Bessel-Hagen, and later independently by other authors, considers gauge symmetries as well when applying Noether's first theorem; this allows for uniquely obtaining the accepted physical energy-momentum tensor in cases such as e.g. electrodynamics — without the need for any ad-hoc "improvement" terms. Given these two distinct methods to obtain an energy-momentum tensor, the question arises as to whether one of these methods corresponds to a preferable application of Noether's first theorem. Using the converse of Noether's first theorem, we show that the Bessel-Hagen type transformations are uniquely selected in the case of electrodynamics — and thus the converse of Noether's first theorem powerfully dissolves the methodological ambiguity at hand. We then go on to discuss further ambiguity issues with respect to energy-momentum tensors in spin-2 theory that can be addressed via Noether's converse. Finally, we put the search for proper Noether energy-momentum tensors into context with recent claims that Noether's theorem and its converse make statements on equivalence classes of symmetries and conservation laws.

2.1.1 Introduction

Physicists have long exploited symmetries to simplify problems. In Lagrangian mechanics, cyclic coordinates (that is, generalized coordinates q_i such that $\partial\mathcal{L}/\partial q_i = 0$ for the Lagrangian $\mathcal{L}$) signal the presence of a symmetry, and the Euler-Lagrange equations imply that the associated conjugate momenta p_i are conserved.[1] It is hard to understate the practical importance of finding conserved quantities, thereby reducing the number of variables and making it much easier to find solutions. Noether's celebrated (1918) paper significantly clarified the mathematical structure underlying these earlier results.[2] The rich line of work stemming from her seminal contribution has elucidated three intertwined aspects of physical theories: laws, symmetries, and conservation principles.

Conventional wisdom now holds that Noether's theorem and its converse universally link a certain kind of continuous symmetry (such as Poincaré translation) to a certain kind of conserved current (such as the energy-momentum tensor).[3] Although based on a kernel of truth, this conventional wisdom reflects an overly simplified picture of the mathematical physics. As a starting point for the discussion below, consider the following specific claim often taken to follow from the Noether machinery: a subset of the variational symmetries of the action, namely spatial and temporal translations, are associated with energy-momentum conservation. Here we encounter an immediate difficulty: applying Noether's first theorem in the context of field theory (as described in §2.1.3), the 4-parameter translation subgroup of the Poincaré group yields what is called the "canonical Noether energy-momentum tensor" ($T_C^{\mu\nu}$). For most classical field theories, the canonical tensor lacks features required for a physically sensible energy-momentum tensor, and differs from known physical energy-momentum tensors established in other ways.[4] Such results raise two related challenges to the conventional wisdom: do the quantities that actually follow from applying Noether's theorem have a clear physical interpretation, and does Noether's theorem need to be supplemented in order to derive physically meaningful conserved quantities? Particularly striking is the existence of inequivalent definitions of the energy-momentum tensor, a central physical quantity in any classical field theory.[5]

Typical textbook presentations leave the impression that Noether's theorem fails to yield the correct energy-momentum tensor. They mention the unappealing features of $T_C^{\mu\nu}$, and then immediately propose a fix. Such fixes amount to variations on a theme going back to [23], who added the divergence of a so-called superpotential to $T_C^{\mu\nu}$ such that a new "Belinfante" energy-momentum tensor $T_B^{\mu\nu} = T_C^{\mu\nu} + \partial_\alpha b^{[\mu\alpha]\nu}$ recovers the correct answer for electrodynamics[6] if an

[1] See [44] for a pedagogical presentation.

[2] noether1918, translated into English by [124].

[3] The metaphysical work of [132], for instance, rests on this. Note that the "energy-momentum tensor" is sometimes referred to as "stress-energy tensor".

[4] We will focus on energy-momentum tensors in the ensuing discussion, but similar issues arise for other conserved currents, such as angular momentum tensors.

[5] The inequivalent definitions mentioned here are twofold: (i) there are multiple definitions for deriving an energy-momentum tensor (see [29], [13]), (ii) within certain specific theories there exist multiple inequivalent expressions (such as linearized gravity, see ([28], [8]).

[6] In the following article, when we refer to electrodynamics, it is implied that we refer to sourceless electrody-

on-shell condition is imposed (discussed in more detail below in §2.1.4). This does not follow directly from Noether's theorem itself, suggesting that some form of "improvement" is needed to find physically meaningful conserved quantities.[7] But any such improvement approach has the unsavoury air of devising a series of poorly justified steps to arrive at an answer found in the back of the book. What happens when we do not already know, or have independent ways of finding, the correct form for the energy-momentum tensor?

Thankfully there is another approach, albeit much less common in the literature: we can use the Poincaré translation symmetry *and* gauge symmetries of the action together in Noether's first theorem. As we will see below, in the case of electrodynamics this leads directly to the correct energy-momentum tensor. This was Bessel-Hagen's neglected contribution, inspired in part by Noether herself [26]. By contrast with the "improvement" approach, we will argue that this approach exploits all of the relevant variational symmetries of the action in applying Noether's theorem. We will argue for the superiority of this approach based on using the converse of Noether's first theorem, which correctly identifies the proper variational symmetries (those derived using Method II in §2.1.4) of the Lagrangian from the accepted form of the energy-momentum tensor. The derivation of the canonical stress-energy tensor fails to use the full power of the mathematical machinery that Noether has given us by considering only a restricted subset of the variational symmetries. Thus, at least in the context of Lagrangian field theories in flat spacetime, the conventional wisdom of a universal linkage between symmetries and conservation laws can be refined to that of a linkage between a specific variational symmetry — to be introduced below — and the set of physical conservation laws for the theory.

This line of argument does not address Lagrangian field theories in curved spacetime, which lack the global symmetries needed to obtain the energy-momentum tensor via Noether's theorem. Physicists then typically use Hilbert's definition of the energy-momentum tensor $T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}$, which is sometimes referred to as the metric energy-momentum tensor. The Hilbert energy-momentum tensor is by definition symmetric, thereby avoiding one of the major flaws of the canonical Noether tensor. From this expression, the Hilbert energy-momentum tensor in Minkowski spacetime is defined as the curved spacetime Hilbert energy-momentum tensor with all metric tensors $g_{\mu\nu}$ replaced by the Minkowski metric $\eta_{\mu\nu}$. [29] outlines the distinction between the curved spacetime and Minkowski spacetime definitions of the Hilbert tensor.[8] In any case, our discussion will focus on the status of the energy-momentum ten-

namics.

[7] There are a number of proposals regarding how to "improve" the energy-momentum tensor in the literature; see forger2004,blaschke2016 for recent surveys.

[8] The Hilbert energy-momentum tensor in Minkowski spacetime ($T_{H,\eta}^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}\Big|_{g=\eta}$) is not, in general, equivalent to energy-momentum tensors derived via Noether's theorem from the 4-parameter Poincare translation, see [13].

sor and associated conservation laws in flat (Minkowski) spacetime — we will consider only energy-momentum tensors which are derived from Noether's first theorem in this article.

The plan for the paper is as follows. This section concludes with a tribute to Bessel-Hagen. We then briefly introduce the energy-momentum tensor, and review properties required for it to be regarded as physically reasonable. In the next two sections, we use classical electrodynamics as a simple case to introduce Noether's first theorem and its subtleties. Section 2.1.3 introduces Noether's first theorem. Section 2.1.4 considers the two approaches described above for defining Noether currents, focusing on the energy-momentum tensor: Method I uses translation symmetries to yield the canonical energy-momentum tensor, which requires "improvement" terms to yield the correct energy-momentum tensor; Method II, by contrast, considers a broader class of variational symmetries and leads directly to the correct energy-momentum tensor. We then argue in favour of the second method based on the converse of Noether's theorem. Section 2.1.5 considers how this line of argument applies to a variety of other cases, including in particular the challenge of defining an energy-momentum tensor for the gravitational field in linearised gravity. Section 2.1.6 brings out one of the themes running through the discussion, namely the challenge of tracking the physical significance of these structural properties of field theories. There are several steps in the early sections where it is tempting to describe both symmetries and conserved currents only up to an equivalence class. We aim to identify clearly the limitations of this move, and develop our position by contrast with recent philosophical discussions about how symmetries relate to the representational capacities of our theories (considering, in particular, Brown's contribution in this volume). Finally we discuss the outlook and conclusions of our work in section 2.1.7.

An ode to Bessel-Hagen

Alongside these systematic aims, we want to use the occasion to clarify the contribution of Erich Bessel-Hagen (1898 - 1946) to the Noether machinery. On the one hand, Bessel-Hagen seems to be often wrongly treated as the *sole* originator of the generalisation of Noether's theorem to invariance under symmetry transformations *up to divergence*.[9] Yet, as [124] (see her section 4.2) notes, Bessel-Hagen himself acknowledges his debt to Noether herself (to a certain degree, at least):

> Zuerst gebe ich die beiden E. Noetherschen Sätze an, und zwar in einer etwas allgemeineren Fassung als sie in der zitierten Note stehen. Ich verdanke diese einer mündlichen Mitteilung von Fräulein Emmy Noether selbst. ([26], p. 260)[10]

[9] Also in the philosophy literature, see for instance, [37].

[10] I first present the two Noetherian propositions, albeit in a slightly more general fashion than they can be

On the other hand, Bessel-Hagen does not seem to be widely known for his central contribution in that very same paper, namely the introduction of what we call Method II: the application of Noether's first theorem in light of gauge symmetries when deriving the complete set of conformal conservation laws for classical electrodynamics. Bessel-Hagen's work has been independently reproduced — by, among others, [70, 153]; the reception of his paper in the English speaking world, however, suffered from the fact that a translation first appeared in 2006 [108], arguably much too late. Even though the method has resurfaced in some textbooks as well [39, 177], it remains relatively unknown in the wider physics literature.

We note that application of the Bessel-Hagen method to a wider class of special relativistic field theories is the topic of an upcoming article [12]. In the present article, the focus on lies on energy-momentum tensors and classical electrodynamics in contrast with the more common canonical Noether approach. A central goal of this article is to use the converse of Noether's first theorem to solve this methodological ambiguity, which as we will see, uniquely specifies the variational symmetries first derived in [26].

2.1.2 Energy-Momentum Tensors

Einstein took the general formulation of conservation laws in terms of the energy-momentum tensor to be "the most important new advance in the theory of relativity" (as of 1912). The energy-momentum tensor has a central role in the new conception of mechanics and field theory, as Einstein went on to emphasize:[11]

> To every kind of material process we want to study, we have to assign a symmetric tensor $T_{\mu\nu}$ [...] The problem to be solved always consists in finding out how $T_{\mu\nu}$ is to be formed from the variables characterizing the processes under consideration. If several processes can be isolated in the energy-momentum balance that take place in the same region, we have to assign to each individual process its own stress-energy tensor ($T^1_{\mu\nu}$, and etc.) and set $T_{\mu\nu}$ equal to the sum of these individual tensors. (CPAE Vol. 4, Doc. 1, [p. 63])

Strikingly, Einstein treats all "material processes," whether they involve electromagnetic fields or matter as described by continuum mechanics, as on a par: the fundamental dynamical quantity in each case is the energy-momentum tensor. How then are we to find an appropriate $T_{\mu\nu}$ for various processes we aim to describe?

found in the cited note. I owe these propositions to an oral communication by Miss Emmy Noether herself. (Own translation)

[11] See [113] for an insightful discussion of the importance of the energy-momentum tensor in the transition to relativistic mechanics.

Before turning to that question, recall that the energy-momentum tensor (also known as the stress-energy tensor) encodes information regarding energy-momentum densities and fluxes for different kinds of "material processes." In relativistic mechanics this is all captured in a single rank-two tensor, $T^{\mu\nu}$: the T^{00} component represents energy density, the T^{0i} and T^{i0} components represent energy and momentum flux, respectively, and the T^{ij} components represent stresses (where $i, j = 1, 2, 3$).

As an illustration, the energy-momentum tensor for electromagnetism provides a compact summary of familiar facts about the electromagnetic field. Minkowski formulated electromagnetism in terms of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the vector potential. The energy-momentum tensor takes the following form:[12]

$$T^{\mu\nu} = F^{\mu\alpha}F^{\nu}_{\ \alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.1}$$

The invariance of the field tensor under gauge transformations ($A'_\mu = A_\mu + \partial_\mu\phi$, for a scalar ϕ) implies gauge invariance of $T^{\mu\nu}$. More generally, we require gauge invariance of $T^{\mu\nu}$ because the energy-momentum tensor represents observable quantities directly. In this case, we have constructed the energy-momentum tensor based on what we already know about the relevant field. Historically, von Laue extended this constructive approach, writing down appropriate energy-momentum tensors for extended stressed bodies, relativistic fluids, and other cases, based on prior knowledge about energy and momentum in each case.

To what extent can we determine the form of $T^{\mu\nu}$ for a new classical field ψ (whether scalar, vector, tensor,...) based on general principles, or on specific features of ψ's dynamics? There are two main types of constraints a tensor would be expected to satisfy to be plausibly interpreted as representing energy-momentum of the field. The first set of constraints stem from the idea that all matter fields "carry positive energy-momentum." More formally, for arbitrary regions of spacetime R, $T^{\mu\nu}$ vanishes on R iff the field ψ vanishes.[13] Further constraints can be imposed to capture the idea that the energy-momentum is positive, and that energy-momentum flows respect the causal structure of relativistic spacetime. One fundamental requirement of this kind is that the energy density (the T^{00} component) is bounded from below, so that the

[12] The energy density of electromagnetic fields is given by $U = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2)$, the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ represents energy flux, and the Maxwell stress tensor σ_{ij} represents stress and momentum fluxes. We can express the energy-momentum tensor in terms of these quantities as follows: $T^{\mu\nu} = \begin{pmatrix} -U & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ -S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ -S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$.

[13] Here we are setting aside fields with negative energy density, for which $T^{\mu\nu}$ could vanish through a cancellation of positive and negative energy densities. This condition has to be formulated with greater care for quantum fields, which necessarily admit negative expectation values for the energy density at spacetime points, but versions of this condition have been proposed for open regions.

field cannot serve as an infinite energy source. Gravity is sensitive to the absolute value of the energy, so that it is meaningful to differentiate positive and negative energies for fields coupled to gravity. Further constraints can then be imposed: the weak energy condition, for example, requires that the energy density (the T^{00} component) is non-negative, as measured by all observers. The dominant energy condition holds if, in addition, momentum fluxes stay within the light cone. There is a long list of other energy conditions that have been used to prove results such as the singularity theorems.[14]

A second set of constraints, and the main focus of the ensuing discussion, regards symmetries and conserved quantities. We will take the satisfaction of an appropriate conservation principle as a defining feature of energy-momentum.[15] Given an appropriate $T^{\mu\nu}$, the on-shell conservation principle can be succinctly stated: $\partial_\mu T^{\mu\nu} = k^\nu$. (For a free field, $k^\nu = 0$, otherwise k^ν represents an external force density.[16]) In classical mechanics, the conservation of energy and momentum stem from space-time translation symmetries, so it is plausible to begin by constructing a tensor combining the conserved currents associated with these symmetries. The variational symmetries of an action S consist of the transformations that leave S invariant, and Noether's first theorem associates a conserved quantity with each element of the finite group of transformations. In electrodynamics these symmetry constraints, the variational symmetries of the action for coordinates (conformal symmetries) and fields (gauge symmetries), are what is required to obtain the known conservation laws, which we detail in §2.1.4; the 15 conservation laws are associated to the (finite) 15 parameter conformal group of transformations.

Other properties of the energy-momentum tensor follow from symmetries of the field theory for specific types of fields. For classical field theories with conformal symmetry, for example, the energy-momentum tensor will be trace-free so that the conformal $C^{\rho\alpha}$ and dilatation D^ρ tensors are conserved.[17]

We take these two types of constraints as requirements that a rank two tensor must satisfy to be a plausible candidate for an "energy-momentum tensor" of particular importance in considering energy-momentum tensors proposed for a new field ψ rather than constructed based on prior knowledge. Perusing the physics literature suggests that these two types of constraints do not suffice to determine a unique choice: there are several proposed, apparently inequivalent, candidates for the "energy-momentum tensor for ψ" (for a variety of different fields). Our overall aim below is to argue against this view. Several of the candidate energy-momentum tensors

[14]See [55] for a comprehensive review of energy conditions and their status.

[15]This leaves open the possibility that there are fields, such as the gravitational field in general relativity, that lack an energy-momentum tensor in this sense.

[16]In the case of electrodynamics sourced by J^α this is the force density $f^\nu = \partial_\mu T^{\mu\nu} = F^\nu{}_\alpha J^\alpha$ which includes the Lorentz force density in the spatial components ($f^i = \partial_\mu T^{\mu i} = J^\rho F^i{}_\rho = \rho\vec{E} + \vec{J}\times\vec{B}$).

[17]In the case of electrodynamics, this statement is directly related to that the associated quantum particles are massless (see [85], p. 563).

are unworthy of the name. Take, for example, the question of whether we should require that the energy-momentum is symmetric under exchange of indices ($T^{\mu\nu} = T^{\nu\mu}$). Failure of this to hold in a mechanical system would lead to torque and the possibility of unlimited angular acceleration.[18] Similar problems arise in field theories. Just as in the case of mechanics, a non symmetric energy-momentum tensor would entail failure of angular momentum conservation. We therefore restrict our attention to symmetric rank-two tensors; this already eliminates many candidates discussed in the literature. We will further argue that we need to take into account more than just spacetime translations in building the energy-momentum tensor out of conserved currents, as we will illustrate next by considering the case of electromagnetism in more detail.

2.1.3 Noether's first theorem for classical electrodynamics

Noether's first theorem, applied to a particular Lagrangian density, yields a relationship between the Euler-Lagrange equation of motion and Noether current of the theory of the form $E^A \delta\phi_A + \partial_\mu J^\mu = 0$, where E^A is the Euler-Lagrange equation for the rank-A field and J^μ is the Noether current. For the Lagrangian density of electrodynamics $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, this takes the form [87]:

$$\left(\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\right)\delta\bar{A}_\nu + \partial_\rho\left(\eta^{\rho\beta}\mathcal{L}\delta x_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\delta\bar{A}_\nu\right) = 0, \tag{2.2}$$

where

$$\delta\bar{A}_\nu = -\partial^\beta A_\nu \delta x_\beta + \delta A_\nu \tag{2.3}$$

is the complete set of symmetry transformations that are linked to Noether's first theorem.[19]

The two methods we will discuss diverge with regard to the general form of the transformations $\delta\bar{A}_\nu$. Consider first the difference between $\delta\bar{A}_\nu$ and δA_ν: the non-bar transformation of fields is the difference in transformed fields as a function of their respective coordinates,

$$\delta A_\nu = A'_\nu(x') - A_\nu(x). \tag{2.4}$$

By contrast, the bar transformations of fields is the difference in transformed fields as a function

[18] An angular momentum tensor $M^{\rho\mu\nu} = x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu}$ has is conserved iff $T^{\mu\nu}$ is symmetric. The angular momentum relative to a given event chosen as an origin can be obtained by integrating $M^{\rho\mu\nu}$; see Chapter 5 of [151] for further discussion of this point, and regarding the general properties of energy-momentum tensors.

[19] Equation (2.2) corresponds to Equation 12 in Noether's paper [124] for the specific Lagrangian of classical electrodynamics, and Equation (2.3) corresponds to Equation 9 in Noether's paper. Our x_β (coordinates) correspond to her independent variables x_n and our A_ν (fields) correspond to her functions of these independent variables u_i.

of the same (non-transformed) coordinates,

$$\delta\bar{A}_\nu = A'_\nu(x) - A_\nu(x), \tag{2.5}$$

where the bar notation is adopted from Noether's paper for this particular transformation in her Equation 9. The following subsections treat the three types of transformations ($\delta\bar{A}_\nu$) relevant to Noether's first theorem in our discussions: (1) the two terms which correspond to the Lie derivative of the fields (canonical and contragredient transformations) and (2) the term which corresponds to gauge symmetries of the action.

Transformations associated to the Lie derivative

In this subsection, we describe two of the contributions to the transformations of fields $\delta\bar{A}_\nu$ arising from infinitesimal change of coordinates δx^ν. These two transformations follow directly from the Lie derivative of the four potential A_ν with respect to the infinitesimal change in coordinates δx,

$$\pounds_{\delta x} A_\nu = -\delta x^\beta \partial_\beta A_\nu - A_\beta \partial_\nu \delta x^\beta = \delta_C A_\nu + \delta_T A_\nu. \tag{2.6}$$

The Lie derivative represents the coordinate invariant change of a tensor field along the flow of a vector field, which is in this case the infinitesimal change in coordinates δx.

We have denoted the two terms in this expression as $\delta_C A_\nu$ and $\delta_T A_\nu$, respectively. The first term, $\delta_C A_\nu = -\delta x^\beta \partial_\beta A_\nu$, is exactly what is found in the first term of (2.3). This term alone is used to derive the canonical Noether energy-momentum tensor when $\delta x^\beta = a^\beta$ is the 4-parameter Poincaré translation, thus we will refer to this as the canonical transformations. The second term, $\delta_T A_\nu = -A_\beta \partial_\nu \delta x^\beta$, we will refer to as contragredient transformations as Bessel-Hagen did in his article; they are associated to the transformation properties of a tensor. This contribution is zero for $\delta x^\beta = a^\beta$, and thus does not factor into energy-momentum tensor discussion. However, for any non-constant δx^β this contribution is nonzero and essential for deriving the associated conserved tensors, such as the angular momentum tensor resulting from the remaining parameters of the Poincaré group.

Canonical transformations and the canonical Noether energy-momentum tensor

If we restrict ourselves to canonical transformations, $-\partial^\beta A_\nu \delta x_\beta$ with $\delta A_\nu = 0$, and no gauge symmetries, we have only

$$\delta\bar{A}_\nu = \delta_C A_\nu = -\partial^\beta A_\nu \delta x_\beta \tag{2.7}$$

to substitute into Noether's first theorem (2.2). We will use $\delta_C A_\nu$ to indicate these canonical transformations. We then obtain:

$$\left(\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\right)\delta\bar{A}_\nu + \partial_\rho\left([\eta^{\rho\beta}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\partial^\beta A_\nu]\delta x_\beta\right) = 0 \tag{2.8}$$

The square brackets contain what is known as the "canonical Noether energy-momentum tensor" $T_C^{\rho\beta}$ for a Lagrangian density of the form $\partial A \partial A$ such as $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ of electrodynamics. In the case of the 4-parameter Poincaré translation $\delta x_\beta = a_\beta$, we can factor out the constant a_β from the divergence yielding $E^\nu \delta\bar{A}_\nu + a_\beta \partial_\rho T_C^{\rho\beta} = 0$, where

$$T_C^{\rho\beta} = \eta^{\rho\beta}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)}\partial^\beta A_\nu \tag{2.9}$$

and E^ν is the Euler-Lagrange equation of motion (in this case, the non-homogeneous Maxwell's equations).

However, substituting the Lagrangian density for electrodynamics $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $\frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\nu)} = -F^{\rho\nu}$, yields

$$T_C^{\rho\beta} = F^{\rho\nu}\partial^\beta A_\nu - \frac{1}{4}\eta^{\rho\beta}F_{\mu\nu}F^{\mu\nu}, \tag{2.10}$$

the canonical Noether energy-momentum tensor for classical electrodynamics. By contrast, the accepted energy-momentum tensor $T^{\mu\nu}$ for the theory is given by (2.1) above.

This specific case illustrates two distinct problems for the canonical energy-momentum tensor that hold more broadly. First, the result simply does not match with an independently motivated expression for the energy-momentum tensor, based on an understanding of energy and momentum densities and fluxes for the relevant field. Second, the canonical tensor lacks essential properties: in general it is neither symmetric, nor gauge invariant, nor trace-free. There are special cases where some of these properties hold. For example, a symmetric tensor follows from (2.9) for a Klein-Gordon scalar field; yet even then, there are alternative tensors which improve on the canonical expression by being trace-free [45]. In some of these cases, it may not be obvious whether the canonical tensor or some other candidate tensor is to be preferred. We do not claim to have a way to resolve this debate across the board; rather, there are several clear cases (like electromagnetism) where the canonical energy-momentum tensor fails to have the right form.

Contragredient transformations

The non-bar transformation of fields δA_ν (second term in (2.3)) is referred to by Bessel-Hagen as being associated to the "contragredient" transformations of the fields, which in current treatments follow simply from the definition of a contravariant tensor (in this case vector):

$$A'^{\nu}(x') = \frac{\partial x'^{\nu}}{\partial x^{\mu}} A^{\mu}(x) \tag{2.11}$$

Inserting this into the contravariant form of (2.4) we have for δA^{ν},

$$\delta A^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} A^{\mu}(x) - A^{\nu}(x). \tag{2.12}$$

If we consider the transformation of coordinates,

$$x'^{\nu} = x^{\nu} + \delta x^{\nu} \tag{2.13}$$

In particular, then, $\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta^{\nu}_{\mu} + \partial_{\mu}(\delta x^{\nu})$. Substituting (2.13) into (2.12) we have $\delta A^{\nu} = A^{\mu}\partial_{\mu}\delta x^{\nu}$. To determine the covariant form of this expression, we can consider the identity $A^{\mu}A_{\mu} = A'^{\mu}A'_{\mu}$ as a function of their respective coordinates, and solve for the transformation $\delta_T A_{\nu}$, which is exactly the contragredient transformation presented by Bessel-Hagen in his equation 18,

$$\delta_T A_{\nu} = -A_{\mu}\partial_{\nu}\delta x^{\mu} \tag{2.14}$$

where $\delta_T A_{\nu}$ indicates that this is the transformation based on the definition of a tensor T. Note that we require the covariant form of this transformation due to our presentation of the Noether identity in (2.2).

For higher rank tensors this contribution can easily become quite complicated. However, in the case of energy-momentum tensor derivation, when we have the 4-parameter Poincaré translation $\delta x^{\mu} = a^{\mu}$ — regardless of Method I or Method II — $\delta_T A_{\nu} = 0$ since a^{μ} is a constant. For this reason, since most of the discussions of conservation laws focus solely on the energy-momentum tensor at the expense of other conserved quantities such as angular momentum, this contribution usually drops out of the picture. (Yet we need the contragredient transformations for the derivation of conservation laws linked to non-constant coordinate symmetries δx^{μ}.)

Gauge (field) symmetries of the action

There is also the possibility of gauge (field) symmetries of the action, often overlooked from the perspective of Noether's first theorem because they are thought to be relevant only to Noether's second theorem. The Bessel-Hagen et al. approach uses these symmetries as well to derive the known conservation laws of electrodynamics directly from Noether's first theorem. Mixing the complete set of coordinate and field symmetries is essential to obtaining the accepted energy-momentum tensor of electrodynamics from Noether's first theorem. To highlight this point we briefly touch on Noether's second theorem, again with a focus on electrodynamics.

The basic idea behind Noether's second theorem is that field (gauge) symmetries that leave the action invariant (in her case, the "infinite continuous group" of transformations of the functions u) can be integrated by parts to remove the derivatives on the field transformations; neglecting boundary terms (instead of keeping them as in the case of the Noether current in the first theorem) results in an identity in terms of the Euler-Lagrange equation. [20]

In the case of electrodynamics, discarding boundary terms (all terms under a total divergence) leaves the standard Euler-Lagrange equation, Maxwell's $\partial_\rho F^{\rho\nu}$, as,

$$\partial_\rho F^{\rho\nu} \delta \bar{A}_\nu = 0 \tag{2.15}$$

Now taking the gauge transformation $A'_\nu = A_\nu + \partial_\nu \phi$ (where ϕ is a scalar), we have,

$$\delta \bar{A}_\nu = \delta_g A_\nu = \partial_\nu \phi \tag{2.16}$$

where we denote $\delta_g A_\nu$ to emphasize the transformation associated to the gauge symmetry of the action. From (2.15) and (2.16) we therefore have $\partial_\rho F^{\rho\nu} \partial_\nu \phi = 0$. Integrating by parts and discarding the resulting boundary term we are left with the well known identity for Noether's second theorem in electrodynamics,

$$\partial_\rho \partial_\nu F^{\rho\nu} \phi = 0 \tag{2.17}$$

and thus $\partial_\rho \partial_\nu F^{\rho\nu} = 0$. It is the incorporation of this transformation (2.16) that is then also essential for directly deriving the complete set of conservation laws from Noether's first theorem, including the accepted energy-momentum tensor (2.1). By use of the converse of Noether's second theorem we have a concrete methodology for obtaining the variational gauge symmetry $\delta \bar{A}_\nu$ that is required for application of Method II — similar to the application of Killing's equation for obtaining the coordinate symmetries required by Noether's first theorem.

Summary

In summary, Noether's first theorem can be used to obtain a relationship between the Euler-Lagrange equation and conservation laws for field theories such as electrodynamics (2.2). The Noether current depends on the coordinate symmetry transformation δx_β and field symmetry transformations $\delta \bar{A}_\nu$; i.e. any symmetry transformation of the action must be introduced through these contributions in order to derive corresponding on-shell conserved currents. We

[20] See Equation 16 in [159] and associated discussion for statements on Noether's second theorem. Notably, the converse also holds, namely that the existence of such identity implies invariance of the action under an infinite continuous group.

distinguished three main types of transformations of fields, which can be simultaneously applied to (2.2) in the form,

$$\delta\bar{A}_\nu = \delta_C A_\nu + \delta_T A_\nu + \delta_g A_\nu \tag{2.18}$$

where we have the canonical transformations (2.7), contragredient transformations (2.14) and gauge transformations (2.16). In the case of electrodynamics, this gives

$$\delta\bar{A}_\nu = -\partial^\mu A_\nu \delta x_\mu - A_\mu \partial_\nu \delta x^\mu + \partial_\nu \phi \tag{2.19}$$

These transformations are the complete set required to derive conservation laws in standard field theories such as electrodynamics. Bessel-Hagen derived all 15 conservation laws of electrodynamics which are associated to the 15 parameter conformal group of infinitesimal coordinate transformations,

$$\delta x_\alpha = a_\alpha + \omega_{\alpha\beta} x^\beta + S\, x_\alpha + 2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu \tag{2.20}$$

In the case of the four-parameter Poincaré translation $\delta x_\beta = a_\beta$, the coordinate symmetry associated to energy-momentum tensor derivation, we have $\delta_T A_\nu = 0$ leaving only two contributions to the transformation of fields $\delta\bar{A}_\nu$.[21]

Notably, the presence of mixed coordinate and field transformations should be no surprise to anyone who has actually read Noether's paper, as she explicitly admits that her first theorem holds for a combination of the symmetries:

> In the case of a "mixed group," if one assumes similarly that Δx and Δu are linear in the ε and the $p(x)$, one sees that, by setting the $p(x)$ and the ε successively equal to zero, divergence relations ... as well as identities .. are satisfied. (translated from [159], p. 243)[22]

where the equation 13 she refers to is identity associated to her first theorem that we consider in this article. It is exactly this freedom that Bessel-Hagen, in fact in consultation with Noether

[21]The term $\omega_{\alpha\beta}x^\beta$, associated to the angular momentum tensor $M^{\rho\alpha\beta} = x^\alpha T^{\rho\beta} - x^\beta T^{\rho\alpha}$, consists of the remaining 6 parameters in the Poincaré group through the antisymmetric parameter $\omega_{\alpha\beta}$. Terms $S\,x_\alpha$ and $2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu$ correspond to the dilatation tensor D^ρ and conformal tensor $C^{\rho\alpha}$. Direct substitution of the various terms in (2.20) into (2.19) give the $\delta\bar{A}_\nu$ which can be directly substituted into (2.2) to derive known physical conservation laws in e.g. electrodynamics. Transformations (2.20) can be found by solving the conformal Killing's equation. See [174] for a self-contained derivation of the conformal invariance of electrodynamics.

[22]"Setzt man entsprechend einer ‚gemischten Gruppe' Δx und Δu linear in den ε und den $p(x)$ an, so sieht man, indem man einmal die $p(x)$, einmal die ε Null setzt, daß sowohl Divergenzrelationen ..., wie Abhängigkeiten ... bestehen."

herself, used to apply the first theorem successfully to electrodynamics; this is the topic of the section on Method II.

In the section on Method I, we will discuss the case when $\delta_g A_\nu = 0$, i.e. when we only use the canonical transformations, yielding the canonical Noether energy-momentum tensor $T_C^{\mu\nu}$. Since this is not the correct energy-momentum tensor $T^{\mu\nu}$ of electrodynamics, various ad-hoc "improvements" have been considered in the literature that add terms to $T_C^{\mu\nu}$ in order to obtain the desired result. In the section on Method II, we will discuss the less common method in the literature, which does not make restrictions on $\delta\bar{A}_\nu$ and keeps the most general (2.18). In this case, the well known $T^{\mu\nu}$ of electrodynamics is directly derived with no ad-hoc "improvements" needed.

2.1.4 A Tale of Two Methods

Method I: Canonical Tensor plus 'Improvements'

In this section, we consider the most common method in the literature for deriving the generally accepted energy-momentum tensor 'from' Noether's first theorem. As emphasized above, the canonical Noether tensor $T_C^{\mu\nu}$ obtained on this approach (2.10) differs from the accepted energy-momentum tensor (2.1). Hence Noether's theorem apparently fails to properly identify the conserved quantities associated with symmetries even in the most familiar case. On our view, this "canonical" result reflects a basic mistake: it does not take into account all of the relevant variational symmetries needed to build an energy-momentum tensor. We will see how to employ Noether's theorem more effectively to do so through what we call Method II in the next section (§2.1.4).

Usually the the canonical energy-momentum tensor is "improved" by adding specific terms (see, e.g., [29]), such as the divergence of a superpotential and terms proportional to the equations of motion. We give an example of this for electrodynamics in this section. One could put the task — a bit provocatively — as follows: Given that $T_C^{\rho\beta}$ is not the result we wanted (or expected), what terms can we add to get the correct answer? Of course this is an ad-hoc approach to fixing the problem, but if it is the best available method we have to obtain the accepted $T^{\rho\beta}$, one might just bite the bullet.[23]

The required "improvement" term in the case of electrodynamics is simply the difference between (2.1) and the canonical expression (2.10),

$$T^{\rho\beta} - T_C^{\rho\beta} = -F^{\rho\nu}\partial_\nu A^\beta. \tag{2.21}$$

[23]It is worth noting that proponents of Method I are usually unaware of the Bessel-Hagen et. al approach.

All of the various improvements for electrodynamics in the literature ultimately need to give us this term on the right hand side, at minimum after imposing on-shell conditions ([76, 29]). The challenge is how to get the correct tensor, by starting from the canonical expression, and adding "improvement" terms through a well-defined procedure. We will briefly discuss the Belinfante improvement procedure since it is by far the most commonly adopted in the literature.[24]

But before doing so, it is worth discussing the general idea of improvement by superpotentials and terms proportional to the equations of motion; together these form the bulk of possible "improvement" terms. Superpotentials have the form $\Psi^{[\rho\alpha]\sigma}$, the divergence of which $\partial_\alpha\Psi^{[\rho\alpha]\sigma}$ can be added to an energy-momentum tensor without affecting on-shell conservation. This is because indices $[\rho\alpha]$ are anti-symmetric; the divergence of the divergence of a superpotential $\partial_\rho\partial_\alpha\Psi^{[\rho\alpha]\sigma}$ is identically zero off-shell. Adding a superpotential to a Noether energy-momentum tensor for a specific Lagrangian does not spoil conservation (the superpotential is conserved as a mathematical identity on its own), yet doing so may lead to an energy-momentum tensor with the required properties. Terms that vanish on-shell, i.e. terms proportional to the equations of motion, can also be added while preserving on-shell equivalence; in practice terms of this type often must also be added to obtain the accepted form of the energy-momentum tensor.

For electrodynamics, the difference between the accepted and canonical energy-momentum tensors is given by (2.21) above. Writing the extra term as the divergence of a superpotential, we have $-F^{\rho\nu}\partial_\nu A^\beta = \partial_\alpha[-F^{\rho\alpha}A^\beta] + A^\beta\partial_\nu F^{\rho\nu}$, where $\Psi^{[\rho\alpha]\sigma} = -F^{\rho\alpha}A^\beta$, and Maxwell's equations $E^\rho = \partial_\nu F^{\nu\rho}$. Thus, we have,

$$T^{\rho\beta} = T_C^{\rho\beta} + \partial_\alpha\Psi^{[\rho\alpha]\beta} - A^\beta E^\rho \tag{2.22}$$

The Belinfante improvement procedure yields exactly the same superpotential; the divergence of this superpotential as well as a term proportional to the equation of motion can be used to recover the accepted energy-momentum tensor in cases such as electrodynamics. Therefore just by knowing (2.1) we know the form of the required (so-called) "improvement" terms. The Belinfante procedure provides a derivation of this superpotential which we will detail in the following subsection.

The Belinfante symmetrization procedure

[24] [23] is commonly cited as the origin of the the Belinfante improvement procedure, which is sometimes referred to as the Belinfante symmetrization procedure. While it is true that he proposed the ad-hoc addition of the divergence of a "superpotential" required to "improve" the canonical Noether tensor for electrodynamics, we note that the more broad ad-hoc "improvement" of energy-momentum tensors in the literature is, perhaps, unfairly associated to his name and outside the scope of his motivations.

Turning an arbitrary tensor into a symmetric tensor is in principle straightforward: decompose the tensor into a symmetric and antisymmetric part, and then add a new contribution to cancel out the antisymmetric part (in this case, from a superpotential). But more interestingly, arguably, [23][25] showed that a suitable superpotential of this kind can be derived — and he argued that it is related to the spin angular-momentum of the model based on common terms.[26] We wish to add the divergence of the Belinfante superpotential $\partial_\alpha b^{[\rho\alpha]\sigma}$ to the canonical expression (2.10) to form the Belinfante tensor $T_B^{\rho\sigma}$,[27]

$$T_B^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha b^{[\rho\alpha]\sigma} \tag{2.23}$$

where the superpotential $b^{[\rho\gamma]\sigma}$ is defined by a combination of the spin angular momentum tensor $S^{\rho[\sigma\gamma]}$ of the form ([23]):

$$b^{[\rho\gamma]\sigma} = \frac{1}{2}(-S^{\rho[\sigma\gamma]} + S^{\gamma[\sigma\rho]} + S^{\sigma[\gamma\rho]}). \tag{2.24}$$

In electrodynamics this contribution is defined as $S^{\gamma[\alpha\beta]} = \frac{\partial\mathcal{L}}{\partial\partial_\gamma A^\mu}[\eta^{\alpha\mu}A^\beta - \eta^{\beta\mu}A^\alpha]$. Therefore we have,

$$S^{\gamma[\alpha\beta]} = -F^{\gamma\mu}[\delta^\alpha_\mu A^\beta - \delta^\beta_\mu A^\alpha]. \tag{2.25}$$

Inserting (3.105) into the Belinfante superpotential (3.104) we have $b^{[\rho\gamma]\sigma} = -F^{\rho\gamma}A^\sigma$. But this is the same superpotential found from (2.22)! Thus for the Belinfante procedure applied to electrodynamics we have,

$$T_B^{\mu\nu} = T_C^{\mu\nu} + \partial_\alpha(F^{\alpha\mu}A^\nu). \tag{2.26}$$

This differs from the accepted energy-momentum tensor (2.1), according to (2.22), by on-shell terms[28].

[25] Through the help of an uncited Dr. Polansky, see [23].

[26] Since the "total" angular momentum tensor $M^{\rho\alpha\beta} = x^\alpha T^{\rho\beta} - x^\beta T^{\rho\alpha}$ is based on the symmetric energy-momentum tensor $T^{\mu\nu}$, decomposing it into parts (e.g. "spin" angular momentum), one straightforwardly can obtain what terms are missing in the canonical Noether tensor $T_C^{\mu\nu}$ from the symmetric $T^{\mu\nu}$ in terms of the corresponding part of the angular momentum: hence the common name of "symmetrization" procedure, and association to spin angular momentum.

[27] We note that the Belinfante tensor $T_B^{\rho\sigma}$ is sometimes referred to as the Belinfante-Rosenfeld tensor, since [175] independently came to some of these results and published them shortly after [22] first presented them. However, Rosenfeld concedes this point in his article stating that repetition of these results from his point of view may still have some utility.

[28] On-shell equivalence of the Belinfante and Hilbert tensors is a well established result [76, 170]. The Hilbert tensor in Minkowski spacetime is the accepted energy-momentum tensor in cases such as electrodynamics (but not in general, see [13]), thus for electrodynamics the Belinfante-Hilbert relationship can be used to obtain the Belinfante superpotential and associated on-shell terms in (2.26) which are required to correctly improve the canonical Noether tensor.

Therefore the accepted energy-momentum tensor (2.1) is related to the Belinfante tensor (2.26) as follows:

$$T^{\mu\nu} = T_B^{\mu\nu} - A^\nu E^\mu. \tag{2.27}$$

The Belinfante prescription alone does not yield the correct expression without adding this additional term ($-A^\nu E^\mu$) proportional to the equations of motion; equivalence to $T_B^{\mu\nu}$ alone can only be established after imposing the on-shell condition $E^\mu = 0$. Note that requiring such an on-shell condition for just formulating the energy-momentum tensor is a severe restriction; in contrast, the Noether energy-momentum tensor directly obtained in Method II can be defined *without* any on-shell condition — only conservation requires imposition of the equations of motion.

So we see that, when interested in symmetric energy-momentum tensors, the Belinfante symmetrization procedure does provide an on-shell procedural fix in e.g. the case of electrodynamics. But it is just a symmetrization procedure; it is not clear how it would, for instance, help to obtain the physical (i.e. also tracefree and gauge-invariant and not just symmetric) energy-momentum tensor in general.[29] The overemphasis on energy-momentum tensor is a limitation of Method I, geared entirely towards just one of the conserved tensors of special relativistic field theory. Method II treats all conserved currents on the same ground; there is no privileging of e.g. energy-momentum, angular momentum, conformal or dilatation tensors relative to one another — all conserved currents follow directly and uniquely from Noether's first theorem.

If we take the most charitable possible view of Method I, that the ad-hoc "improvements" are entirely physically justified and a necessary correction after applying Noether's first theorem, one unavoidable fact remains: the "improvement" procedure still requires on-shell conditions to equate the accepted $T^{\mu\nu}$ to the improved tensor. In Method II, no on-shell conditions are required to obtain the complete set of conservation laws.

Method II: Including Gauge Symmetries

There is another method for deriving the energy-momentum tensor, such that we directly obtain it from Noether's first theorem without requiring any ad-hoc "improvements". Instead of using the restrictive condition of Method I where we only consider the canonical transformations (2.7), we instead use the most general picture of all possible field transformations such as outlined in (2.18). Considering the transformations of the action

$$\delta \bar{A}_\alpha = -F^\nu{}_\alpha \delta x_\nu, \tag{2.28}$$

[29] [29] provide an improvement procedure based on requiring gauge invariance instead of just symmetry.

Bessel-Hagen [26] (as well as [70, 153], among others) derived all 15 accepted conservation laws of electrodynamics directly from Noether's first theorem using (2.28), (2.2) and the sourcefree Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. In other words, (2.28) are what we call proper transformations.

$$E^{\nu}\delta\bar{A}_{\nu} + \partial_{\rho}\left([F^{\rho\nu}F^{\beta}_{\ \nu} - \frac{1}{4}\eta^{\rho\beta}F_{\mu\nu}F^{\mu\nu}]\delta x_{\beta}\right) = 0 \tag{2.29}$$

where E^{ν} is the Euler-Lagrange equation. Immediately Noether's first theorem leads to the physical energy-momentum tensor (2.1) in square brackets.[30] More precisely, for the case of the 4-parameter Poincaré translation, $\delta x_{\beta} = a_{\beta}$ is a constant that can be pulled out of the total divergence leaving the elegant identity,

$$E^{\nu}\delta\bar{A}_{\nu} + a_{\beta}\partial_{\rho}T^{\rho\beta} = 0 \tag{2.30}$$

Immediately we have a compact identity relating the Euler-Lagrange equation E^{ν} and energy-momentum tensor $T^{\rho\beta}$ of electrodynamic theory. The Lorentz force law, Poynting's theorem are all compactly derived alongside Maxwell's equations. This compact identity makes it easy to appreciate the celebrated elegance of Noether's theorems.

Deriving the proper transformations from the Bessel-Hagen method

How can one obtain the proper transformations (2.28) that lead to the physical conservation laws? The various authors [26, 70, 153, 155, 186] that independently came to this conclusion used slightly different rationales, largely to do with requiring gauge invariance of the Noether current or requiring gauge invariance of the transformations themselves. We will follow the Bessel-Hagen approach because he was first to present this result, and took advice from Noether herself on his paper. More explicit application of the Bessel-Hagen method to electrodynamics can be found in [12]. Starting from the general transformations of fields (2.19) in the case of electrodynamics we have,

$$\delta\bar{A}_{\nu} = -\partial^{\beta}A_{\nu}\delta x_{\beta} - A_{\mu}\partial_{\nu}\delta x^{\mu} + \partial_{\nu}\phi \tag{2.31}$$

The question Bessel-Hagen asked is how to derive the parameter ϕ such that we have the unique gauge invariant energy-momentum tensor of (sourcefree) electrodynamics. To do this we substitute $\delta\bar{A}_{\nu}$ into the Noether current (2.2) and solve for ϕ to obtain a current which is

[30] Using (2.29) the 15 conformal conservation laws of electrodynamics are immediately obtained by inserting δx_{β} from (2.20): four from the divergence of the energy-momentum tensor $T^{\rho\beta} = F^{\rho\nu}F^{\beta}_{\ \nu} - \frac{1}{4}\eta^{\rho\beta}F_{\mu\nu}F^{\mu\nu}$, six from the divergence of the angular momentum tensor $M^{\rho\mu\nu} = x^{\mu}T^{\rho\nu} - x^{\nu}T^{\rho\mu}$, one from the divergence of the dilatation tensor $D^{\rho} = T^{\rho\beta}x_{\beta}$ and four from the divergence of the conformal tensor $C^{\rho\alpha} = T^{\rho\beta}(2x_{\beta}x^{\alpha} - \delta^{\alpha}_{\beta}x_{\lambda}x^{\lambda})$

gauge invariant. The ϕ must depend on both the vector potential A_α (in order to obtain a gauge invariant current) and the infinitesimal transformations of coordinates δx^α (in order to factor out the 4 parameter Poincare translation from the current and obtain the energy-momentum tensor). Bessel-Hagen solved for ϕ, obtaining for the gauge parameter $A_\mu \delta x^\mu$, which is the most trivial scalar combination of the required components. Inserting this ϕ into (2.31) and differentiating the third term we have,

$$\delta \bar{A}_\nu = -\delta x_\beta \partial^\beta A_\nu - A_\mu \partial_\nu \delta x^\mu + \delta x_\beta \partial_\nu A^\beta + A_\mu \partial_\nu \delta x^\mu \tag{2.32}$$

Remarkably the second and last terms on the right hand side cancel (those associated to the contragredient transformations) and we are left with exactly $\delta \bar{A}_\nu = -F^\beta{}_\nu \delta x_\beta$ as in (2.28)! Therefore the proper transformations that directly yield the physical conservation laws can be thought of as a mixing of the various symmetries of the action, as opposed to an independent application of symmetries as in the case of the canonical Noether energy-momentum tensor or Noether's second theorem.

We note that the selection of the gauge parameter, while aided by knowledge of the unique gauge invariant energy-momentum tensor in the case of electrodynamics, can be obtained from the Noether current. For this reason the method applies more generally to models where the energy-momentum tensor is not already known. The more general application of Bessel-Hagen to exactly gauge invariant actions in this way is the subject of [12], in which the Bessel-Hagen method has successfully been applied to several field theories such as Yang-Mills, Kalb-Ramond, third rank antisymmetric fields, and linearized Gauss-Bonnet gravity.

The proper form of the transformation was noticed for Yang-Mills theory by [110], without deriving this from a procedure such as Method II. While the vast majority of textbooks give the canonical picture alone, e.g. [39, 177] have noticed the proper transformations and avoided the restrictive canonical presentation. One of our goals in the following is to settle this ambiguity in favor of the proper transformations through appeal to the converse of Noether's first theorem.

Proper transformation as gauge-invariant transformations

We now know that the appropriate choice of $\delta \bar{A}_\alpha = \delta_C A_\alpha + \delta_g A_\alpha$ in the Noether identity for classical electrodynamics directly leads to the accepted energy-momentum tensor, and that the proper transformation can be chosen by solving for a $\delta_g A_\alpha$ that makes the current invariant. We will explore how to justify the specific choice of $\delta_g A_\alpha$ a posteriori via the converse of Noether's first theorem in the next section, i.e. by starting from the accepted energy-momentum tensor.

Before doing so, we want to explore how to motivate the choice of $\delta_g A_\alpha$ other than by solving the Noether identity for $\delta_g A_\alpha$ while requiring that the energy-momentum is gauge-invariant.

To this end, we will consider [70], who argued that gauge invariance of the transformation $\delta\bar{A}_\alpha$ is the property one can use to determine the proper transformation $\delta\bar{A}_\alpha = -F^\nu{}_\alpha\delta x_\nu$ as in (2.28). eriksen1980 starts from the gauge condition in the case of sourcefree electrodynamics, $\delta_g A_\alpha = \partial\chi$ with $\chi = \chi(A)$. The parameter χ is taken to be a arbitrary gauge parameter we must solve for based on the condition that $\delta_g A_\alpha$ must be gauge invariant. By combining the $\delta_T A_\alpha$ (contragredient) and $\delta_g A_\alpha$ (gauge) transformations, the authors find an equation for χ which does not uniquely determine χ; however they choose the "simple" solution that leaves $\delta_g A_\alpha$ as a whole gauge invariant, which is identically $\chi = \delta x^\nu A_\nu$, exactly what was found by Bessel-Hagen!

One could now note how intuitive the requirement of a gauge-invariant transformation is: as long as all expressions in the assumptions of the Noether theorem are gauge-invariant, the resulting energy-momentum tensor should come out as gauge-invariant too. However, there exist cases where the proper transformations that are used to derive the unique energy-momentum tensor for a theory are not themselves gauge invariant, as we will discuss in Section 2.1.5. This indicates limits in the scope of application of eriksen1980's method. We note that the Bessel-Hagen method works more broadly because it treats both the cases where the transformations themselves are gauge invariant, as well as cases where they are not.

Converse of Noether's first theorem as a test for Noetherian currents

We now use the converse of Noether's first theorem relative to the Lagrangian density of electrodynamics and the accepted energy-momentum tensor (2.1) in order to arrive at the relevant variational symmetry linked to this $T^{\mu\nu}$. As we will see, the converse can generally be used to decide whether an energy-momentum tensor can be directly derived from Noether's first theorem — and thus from Method II.

We can derive the form of the transformations $\delta\bar{A}_\nu$ using the converse of Noether's first theorem based on the accepted energy momentum tensor, (2.1), as follows: We start with

$$E^\nu\delta\bar{A}_\nu + a_\nu\partial_\mu T^{\mu\nu} = 0 \tag{2.33}$$

and Noether's first theorem (2.2). Since the 4-parameter Poincaré translation $\delta x_\beta = a_\beta$ is associated to the energy-momentum tensor it follows that $\delta\bar{A}_\nu = U^\beta_\nu a_\beta$, namely the transformation of fields must be proportional to the 4-parameter a_β. Therefore, we must solve for U^β_ν,

$$E^\nu\delta\bar{A}_\nu + a_\beta\partial_\rho\left(-F^{\rho\nu}U^\beta_\nu - \frac{1}{4}\eta^{\rho\beta}F_{\mu\nu}F^{\mu\nu}\right) = 0 \tag{2.34}$$

Subtracting the two equations (2.33) and (2.34) we have,

$$a_\beta \partial_\rho \left(-F^{\rho\nu} U^\beta{}_\nu - \frac{1}{4} \eta^{\rho\beta} F_{\mu\nu} F^{\mu\nu} \right) = a_\beta \partial_\rho \left(F^{\rho\nu} F^\beta{}_\nu - \frac{1}{4} \eta^{\rho\beta} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.35}$$

We have $U^\beta_\nu = -F^\beta{}_\nu$, and thereby recover the proper transformations for the Poincaré translation $\delta \bar{A}_\nu = -F^\beta{}_\nu a_\beta$. More generally we can solve for δx_ν from the Noether identity and again we have the requirement $\delta \bar{A}_\alpha = -F^\nu{}_\alpha \delta x_\nu$. Thus, if we consider the converse of Noether's first theorem on the accepted $T^{\mu\nu}$ (2.1), the canonical transformation (2.7) associated to the canonical Noether energy-momentum tensor (2.10) never appears in isolation! In other words, in the case of electrodynamics, the converse of Noether's first theorem confirms Method II as the method directly associated to Noether's first theorem. At the same time, we can now realise from the vantage point of the converse Noether theorem that Method I simply uses the wrong symmetry to begin with. The failure to recognise the properly adapted symmetry transformations leads to the need to introduce — and justify, if possible — ad hoc "improvements".

The lesson from electrodynamics generalises: Given a proposed energy-momentum tensor, we learn through Noether's converse which (if any) symmetries are linked to it; if there are none linked to it, then the energy-momentum tensor cannot be derived directly from Noether's first theorem.[31] So, in cases like electrodynamics for which the canonical energy momentum tensor lacks essential properties, we find that the improvements can be avoided by using Noether's first theorem properly (that is, by exploiting the complete set of variational symmetries of the action) — and thus that there is nothing wrong with the Noether method to begin with. In cases where the converse does not give symmetries linked to an energy-momentum tensor, we at least learn that this energy-momentum tensor cannot be derived from Noether's first theorem.

We should acknowledge that the approach we are advocating is not as straightforward in case we do not already know the appropriate energy-momentum tensor for the relevant fields. We have criticized the improvement approach because it apparently relies on knowing the proper form of the energy-momentum tensor in order to find the appropriate improvement terms; yet we also need to know the proper form of the energy-momentum tensor to apply the converse. Although our exposition of Method II was based on a case where we do know the energy-momentum tensor, we claim more generally that it gives a clearer account of the relationship between symmetries and conserved currents. This can lead to a kind of reflective equilibrium in assessing candidates for energy-momentum tensors and the associated symmetries. We will turn to just such a case in the next section, namely linearized spin-2 fields where numerous energy-momentum tensors have been proposed.

[31] This raises the question whether such an object earns the title of energy-momentum tensor in the first place.

2.1.5 Applications beyond electrodynamics

Up to this point we have used electrodynamics to explicate Method II — but it has much broader scope than this. As a matter of fact, a recent series of papers has shown how Method II applies to several classical, relativistic field theories, such as:

- *(Source-free) Yang Mills* [12], with the Lagrangian $\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$. Applying Method II with the "mixed" variational symmetry $\bar{\delta}A^a_\mu = -F^a_{\mu\nu}\delta x^\nu$ (with $\delta x^\nu = a^\nu$) leads to the energy-momentum tensor $T^{\mu\nu} = F^{\mu\lambda}_a F^{a\nu}{}_\lambda - \frac{1}{4}\eta^{\mu\nu}F^a_{\lambda\rho}F^{\lambda\rho}_a$.[32] The energy-momentum tensor is invariant under the gauge transformation $\delta_g \bar{A}_{a\mu} = \partial_\mu\theta_a + C_{abc}A^b_\mu\theta^c$.

- *Linearized Gauss-Bonnet gravity* [14, 9], with the Lagrangian $\mathcal{L} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$.[33] The variational symmetry $\delta\bar{h}_{\rho\sigma} = -2\Gamma^\nu{}_{\rho\sigma}\delta x_\nu$ (with $\delta x^\nu = a^\nu$) leads to the generally accepted energy-momentum tensor,[34] which is gauge-invariant under the spin-2 gauge transformation $\delta_g\bar{h}_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$.

Just as with electrodynamics, applying the mixed variational symmetry in each of these cases leads directly to the accepted energy-momentum tensor. bakerthesisBHpaper discusses several other cases as well.

What can then be said about the scope of the method? For a given gauge invariant Lagrangian density, an exact variational symmetry can be found such that the Noether current associated to the Poincaré translation will be the physical energy-momentum tensor. Notably, it is *not* a necessary criterion that the total symmetry transformation is gauge-invariant: Recalling 2.1.4, the decisive symmetry transformation in electrodynamics (as given by the standard Lagrangian) is gauge-invariant, but the symmetry transformation is not always itself gauge invariant (e.g. linearized Gauss-Bonnet gravity). This means that the proper variational symmetries can not always be systematically obtained from requiring gauge-invariance of the proper transformation, which showcases the restrictions of the procedure presented in section 2.1.4. In other words, as an exactly gauge invariant symmetry transformation will only be sufficient but not necessary for obtaining a gauge-invariant conserved current; Method II à la Bessel-Hagen has a much wider scope than the method of [70].

Put the other way around, problems arise for Method II when: (1) we have a model that does not have an exactly gauge invariant action, so that solving for the right gauge-transformation becomes extremely laborious if not impossible, or (2) when the energy-momentum tensor is

[32] The field strength tensor is given by $F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + C_{abc}A^b_\mu A^c_\nu$ where C_{abc} is the totally antisymmetric structure constant.

[33] Appearing in the Lagrangian are the linearized Riemannian tensor, defined as $R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha})$, and contractions of it.

[34] Explicitly, $T^{\omega\nu} = -R^{\omega\rho\lambda\sigma}R^\nu{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^\nu{}_\lambda - RR^{\nu\omega} + \frac{1}{4}\eta^{\omega\nu}(R_{\mu\lambda\alpha\beta}R^{\mu\lambda\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$

questionable but the BH not applicable because there is just no gauge symmetry to begin with. We will outline now how in both cases at least the Noetherian nature of the energy-momentum tensor can be checked upon application of Noether's converse.

With respect to (2), an interesting application of Noether's converse is to reveal the improved Callan-Coleman-Jackiw (CCJ) traceless energy-momentum tensor of the Klein-Gordon theory (see [45]) as non-Noetherian relative to the standard Lagrangian.[35] However, an energy-momentum tensor that is non-Noetherian relative to some Lagrangian may be Noetherian to the same Lagrangian up to a divergence term; this is exactly the case of the CCJ-energy-momentum tensor (see [127]).

Turning to (1), an appealing possible application of Noether's converse regards the case of the spin-2 Fierz-Pauli action where the gauge symmetry of the equation of motion (linearized diffeomorphisms) is not an exact symmetry of the action; the action is only invariant up to a boundary term (see [14]). Furthermore, there is no gauge-invariant energy-momentum tensor for spin-2 gravity (see [140]) to begin with, so we cannot use gauge invariance to help pick out a unique expression.[36] If an action is not exactly invariant such as in the case of spin-2, generalizations of Noether's theorem to symmetries up to boundary terms (i.e. the non-exact symmetries method in [26]) must be applied; the application of these methods to spin-2 Fierz-Pauli theory is the subject of future work.

To elaborate a bit on the issue: For linearized gravity (massless spin-2 gravity), there are numerous proposals for $T^{\mu\nu}$ (see [28] for an overview). This ambiguity cannot be avoided in, for example, attempts to derive general relativity from a spin-2 field theory that proceed by taking the spin-2 field $h_{\mu\nu}$ to be self-coupled. Which $T^{\mu\nu}$ should be added to the action to represent this self-coupling? Here authors disagree on whether the Einstein field equations can be derived from spin-2 Fierz-Pauli theory, to a large degree based on their choice of which $T^{\mu\nu}$ to select (if they even grant that it is physically well-defined despite its inevitable gauge-dependent nature). (See [163] for a criticism, and [16] for a defense, of conventional wisdom on this issue.)

Linearised (massless) spin-2 gravity is given by the (massless) Fierz-Pauli Lagrangian density

$$\mathcal{L}_{FP} = \frac{1}{4}\left(\partial_\alpha h^\beta_\beta \partial^\alpha h^\gamma_\gamma - \partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} + 2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha} - 2\partial^\alpha h^\beta_\beta \partial^\gamma h_{\gamma\alpha}\right)$$

with canonical Noether energy-momentum tensor $T^{\mu\nu}_C = \eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}\partial^\nu h_{\alpha\beta}$ — the linearized

[35] We call an energy-momentum tensor Noetherian relative to $\mathcal{L}$ if it is directly derivable from a variational symmetry of $\mathcal{L}$ via Noether's first theorem.

[36] Arguably, this jeopardises the application of BH method which is centrally about achieving a gauge-invariant current.

Einstein energy-momentum tensor gives exactly this canonical expression ([184]). Strikingly, this tensor is neither gauge-invariant nor symmetric nor traceless. As we know from [140], there is no energy-momentum tensor for spin-2 Fierz-Pauli theory that is gauge-invariant. Candidates for such "improved" energy-momentum tensors for linearized gravity usually presented include, for example, the linearized Hilbert and Landau-Lifshitz expressions — both of which can be obtained by adding the appropriate divergence of superpotential and terms proportional to the equations of motion to $T_C^{\mu\nu}$ (see [8]). Strong adherents to Method I might then suggest that since any such energy-momentum tensor (conserved on-shell using the spin-2 Fierz-Pauli equation of motion) follows from the addition of improvement terms, that all such linearized gravity energy-momentum tensors are in some sense connected to Noether's first theorem. In [8] it is shown that there are infinitely many such "improved" energy-momentum tensors for linearized gravity.

A much more straightforward approach for spin-2 linearized gravity then is to apply the converse of Noether's first theorem to the various expressions in the literature as we did for electrodynamics in the previous section. This would concretely determine which (if any) can yield δX_β and $\delta\bar{h}_{\mu\nu}$ symmetry transformations to prove a direct and meaningful connection to Noether's first theorem. Concretely, one would have to use the Noether identity

$$\left(\frac{\partial\mathcal{L}}{\partial h_{\mu\nu}} - \partial_\rho\frac{\partial\mathcal{L}}{\partial(\partial_\rho h_{\mu\nu})}\right)\delta\bar{h}_{\mu\nu} + \partial_\rho\left(\eta^{\rho\beta}\mathcal{L}\delta X_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\rho h_{\mu\nu})}\delta\bar{h}_{\mu\nu}\right) = 0 \tag{2.36}$$

for each energy-momentum tensor and solve for $\delta\bar{h}_{\mu\nu}$ in the same way we solved for the variational symmetries of electrodynamics in Section 2.1.4, where $\delta\bar{h}_{\mu\nu} = -\partial^\beta h_{\mu\nu}\delta X_\beta + \delta h_{\mu\nu}$. As in the case of electrodynamics, the term proportional to the Lagrangian density ($\eta^{\rho\beta}\mathcal{L}$) must have a Lagrangian density $\mathcal{L}$ which yields the spin-2 equation of motion in the Euler-Lagrange equation — otherwise the energy-momentum tensor in question will not be associated to spin-2 Fierz-Pauli theory in the context of Noether's first theorem, regardless of the transformations we consider. Regardless of the outcome of this calculation we will have a strong statement about the energy-momentum tensors for spin-2 theory in the literature: either that Noether transformations can select a preferred expression, confirm numerous expressions can be obtained from the Noether approach, or show that there is problems with applying Noether's first theorem to this model as a whole. If transformations can uniquely be solved for by the converse of Noether's first theorem, then we can say that a given expression can be directly derived. If there is no solution, then a given expression cannot be claimed to be associated to Noether's first theorem for a particular Lagrangian density. Whether or not the various published expressions can be directly obtained, regardless of outcome, will provide clear insight into the relationship between the linearized gravity energy-momentum tensors in the literature and Noether's first theorem. We note there are debates as to whether or not such programs are even related to GR (see, for instance, [173, 65]), but regardless this is an active topic that leads to considerable confusion in both the physics and philosophy literature. This application of the converse of Noether's first theorem to spin-2 is the subject of future work.

2.1.6 Equivalence classes

Mathematics often draws finer distinctions than physics requires. Physicists typically treat a unique definition of a fundamental quantity as necessary for understanding its physical significance, and this can conflict with the embarrassment of riches resulting from mathematicians' drive to generalize. A natural response is to regard some range of mathematically distinguished possibilities as falling within an equivalence class, such that physical interpretations need not draw distinctions among its members. Recent philosophical discussions (see [36] in this volume) have argued that Noether's theorem should be read as relating an equivalence class of symmetries to an equivalence class of conservation laws. This suggests that the search for a unique energy-momentum tensor we have been pursuing is misguided or unnecessary — we should be satisfied with an equivalence class of energy-momentum tensors. In this section we aim to adjudicate these questions regarding uniqueness and the appropriate criteria of equivalence, or (perhaps more accurately) at least to survey some of the considerations that bear on them.

There are two rather obvious notions of equivalence related to the results above then: First, in the Noether identity (including vector field theories (2.2) such as electrodynamics), the 'inside' of the divergence on the right-hand side is only determined up to a term that disappears identically when hit by a divergence. In other words, whatever is inside the divergence term, it is only characterised by the Noether identity up to the divergence of a superpotential and terms proportional to the equations of motion. This means that Noether's theorem can be said to only link a variational symmetry to an equivalence class of conservation laws in the sense that two conservation laws are equivalent iff they differ by these superpotential terms (call this *superpotential equivalence*).[37] Secondly, going beyond mere formal considerations, a straightforward empiricist is naturally inclined to only take the divergences of the energy-momentum tensor to be relevant: as you can only measure out solutions — i.e. what arises on-shell —, all empirical data according to the straightforward empiricist is in the solutions; alternatively, this data can be seen as codified into the dynamical equations and the space of possible initial conditions (provided the problem is well-posed). On such a view, then, it is just consequential to define two energy-momentum tensors to be equivalent iff they differ by superpotential terms

[37] Cf. [36].

and terms proportional to the equations of motion (call this *empiricist equivalence*).[38]

So far so good. But the evaluative question simply is: *should* we indeed treat the symmetries as determining only an equivalence class of energy-momentum tensors in the superpotential or even the empiricist sense? In electrodynamics, the empiricist equivalence class would consist of tensors generated from the canonical tensor (2.9) by adding a general linear system of superpotentials and terms proportional to the equations of motion:

$$T^{\mu\nu} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + C_1\partial_\alpha\Psi^{[\mu\alpha]\nu} + C_2A^\nu E^\mu + C_3A^\mu E^\nu + C_4\eta^{\mu\nu}A_\alpha E^\alpha \tag{2.37}$$

where $\Psi^{[\rho\alpha]\sigma}$ is the most general rank-three tensor defined in terms of the potential and derivatives of the potential, with the required symmetry properties. We recover the standard expression by setting $C_n = 0$ (for all n). But even allowing C_n to take arbitrary values, (2.37) will still satisfy the Noether identity (2.2). In the case of spin-2, any of the published energy-momentum tensors can be obtained from the canonical Noether tensor by adding superpotential terms and on-shell contributions [8], leading to a collection of infinitely many energy-momentum tensors in an equivalence class.

A positive answer to the evaluative question would contrast sharply with the common practice in physics of taking (2.1) as the "correct" expression. One possibility is that physicists simply choose one element of the equivalence class by convention or as a matter of convenience. (If the elements of the equivalence class truly "represent the same physical situation," it would be a mistake to demand physical justification of the choice.) But this is not the position one finds; instead, there are various claims to have established the uniquely correct physical expression for the energy-momentum tensor for various classical field theories [95, 76].

We can offer a more insightful explanation as to why this evaluative question should indeed be answered in the negative by the practitioner than a mere conventionalist stance. First, simply note that what we dubbed the empiricist equivalence builds on a heavily local sense of what is empirically relevant: off-shell differences may not be empirically relevant classically but they do for instance become empirically relevant upon quantisation (think of the Feynman path integral picture). Furthermore, in the specific context of energy-momentum tensor, one might gesture at that gravity does care about the absolute value of matter energy-momentum and that we should thus do so then even outside of gravitational theories.

But much more importantly, and independently of empirical matters, it is simply a fact that

[38] To be clear: all of the thus equivalent energy-momentum tensors can be determined empirically (The components of the energy-momentum tensors are functions of field values; so as long as one can measure out field values — which is beyond doubt for classical field theories — the energy-momentum tensors can be measured out.) But none of the thus equivalent energy-momentum tensors can be preferred on empirical grounds according to the local empiricist.

physical practice centrally relies on the adherence to non-empirical theoretical virtues — including the imposition of guiding principles in order to comply with these virtues.[39] From a methodological point of view not only is it thus straightforward to reject the previously introduced notion of empiricist equivalence — even the purely formally implied equivalence up to superpotential in light of the Noether identity has to be dismissed as too coarse-grained. (In practice, accepted physical conservation laws of standard field theories such as electrodynamics and Yang-Mills theory are derived without requiring any notion of equivalence class. What purpose does considering the equivalence class that satisfies the Noether identity serve if the complete set of equations for the theory have already been uniquely obtained?)

The detailed argument runs as follows: a central methodological guiding principle in physical theory construction is that theoretical objects should contain as much information about the theoretical context at play as possible (informativeness principle).[40] This guiding principle can be seen as grounded in different virtues: most obviously, it manages to maximise the information content of the theory in its own right. It may also be seen as realising — possibly as a by-product but this is not clear — other virtues such as the already hinted at connectability to successor theories. In any case, given this principle, it is out of question to consider two energy-momentum tensors as equivalent *tout court* just because they are dynamically equivalent (in the sense that they are equal on-shell). More than that: it is also ruled out now to declare energy-momentum tensors equivalent when only differing by rather trivial (since arbitrary) superpotential and on-shell terms as they change the components of the energy-momentum tensor itself.

So, if we are committed to the informativeness principle — as we take practitioners to be — Noether's theorem only seems to link specific symmetries to specific conservation laws — and not to a whole class of conservation laws. It should be clear by now then that our points in the previous sections about the relation between symmetries and conservation laws (say relative to Method 1 and Method 2) are exactly methodological points under the assumption of the informativeness principle.

2.1.7 Conclusions

Noether's first theorem is one of the most celebrated results in physics. Yet, standard textbook and literature presentation gives the picture that this method fails to derive standard physical conservation laws: the canonical Noether energy-momentum tensor, which is derived using a restricted condition placed on Noether's first theorem, does not give the known physical energy-

[39] For a metaphysical reading of off-shell conservation expressions, see [135].

[40] What we call informativeness principle, might also be re-cast as a principle of theoretical parsimony in the sense that the available structure should be maximally informative so that no surplus structure has to be added.

momentum tensor in foundational models such as electrodynamics and Yang-Mills theory. All of this creates the impression that Noether's first theorem, despite frequent praise in the scientific community, is in some sense not working in practice for our most significant theories. We hope that our presentation of the Bessel-Hagen method (Method II) has let the reader regain confidence in the power of Noether's first theorem when applied to exactly gauge-invariant field theories: using the complete set of (mixed) symmetries of the action (both gauge and coordinate symmetries), one obtains transformations that directly yield the known physical energy-momentum tensor of electrodynamics and theories with a gauge-invariant Lagrangian density more generally. No "improvement" of the energy-momentum tensor is needed to supplement (nor actually advised for by) the Noether machinery.

In showcasing the proper application of Noether's theorem in the context of exactly gauge-invariant theories, we have, moreover, learned that the conventional wisdom that a specific variational symmetry (namely the canonical variational symmetry) is linked to a specific conservation law by the Noether machinery after all remains true within the bulk of classical field theory in practice. As we had already said in the introduction, one is free to question the linkage by more theoretical counterexamples — but this is a question for another day. Yet another interesting insight was gained along the way: contra common characterisations in the literature, Noether's first theorem is not solely concerned with what we called canonical variational symmetries exclusively but rather the complete set of symmetries of the action (this includes gauge symmetries which are sometimes portrayed as being only associated to Noether's second theorem). We have used the converse of Noether's first theorem as a method for emphasizing this fact, as the canonical variational symmetries do not follow from the converse theorem for the majority of accepted physical energy-momentum tensors in the literature.

Finally, there is a sense in which our overall message in favour of Method II could be made even more strongly: Throughout the article, we had tacitly accepted the common theme in the literature to pay special attention to the energy-momentum tensor over and above other conserved currents in special relativistic field theory. This is important to note as it is quite possible that many of the non-uniqueness and ambiguity problems associated to tensor conservation laws are a result of limiting oneself to the case of energy-momentum tensor specifically, and that the large variety of methods for energy-momentum construction compared to the other tensors is rather an issue that may not be solved by treating the energy-momentum tensor as a privileged standalone object. From the point of view of Noether's first theorem and Method II, none of the standard conserved tensors (energy-momentum, angular momentum, conformal and dilatation) are privileged compared to each other. Thus, if we agreed to consider only methodology which links all of the conserved tensors of a theory to variational symmetries simultaneously, Noether's first theorem in the sense of Method II may give a much needed uniqueness in methodology akin to the Euler-Lagrange equation for an equation of motion. Such a view, if adopted, has promise to end the various ambiguity and non-uniqueness problems associated to the energy-momentum tensor once and for all.

2.2 Contemporary controversy with improving the canonical Noether energy-momentum tensor solved one hundred years ago by Bessel-Hagen

2.2.1 Method II (the Bessel-Hagen method) Applied to Gauge Theories

In the previous article we discussed the problem with multiple distinct methods of applying Noether's first theorem to classical gauge theories, and argued that Method II (the Bessel-Hagen method) is the proper application of Noether's first theorem to physical theories based on various factors (off-shell equivalence, no ad-hoc "improvments" required, the converse of Noether's first theorem, etc.). In Bessel-Hagen's 1921 paper [26], he only performed these calculations for classical electrodynamics. Other similar approaches have considered Yang-Mills theory [153], but formal application of the Bessel-Hagen method to various gauge theories beyond electrodynamics have not been considered. In this article we will consider several applications of the Bessel-Hagen method, which verify its applicability to various completely gauge invariant actions. We give the complete set of conformal conservation laws associated to each model we consider, and discuss these results at the end.

This article is unique compared to the others in the PhD thesis, because what is included in the PhD thesis is small fraction of the article which we have prepared to submit to a journal in the near future (it is one of the two articles in this thesis that are not currently published). The reason for this is that this article (in complete form) is a semi-historical, semi-review, semi-philosophical, semi-mathematical and semi-physical 50+ page paper which covers the history of Noether's first theorem and the controversy around improving the canonical Noether energy-momentum tensor, the Bessel-Hagen method, and associated conflicting results. These historical, review, and philosophical aspects have been removed from the article included in the PhD thesis for brevity (the thesis is already over 200 pages without this article included) and because in the previous article [15] in Section 2.1 there is a lot of overlap which covers the general introductory material required to understand what we include here. What we include here is simply the application of Bessel-Hagen method to three models: Kalb-Ramond, Yang-Mills, and totally antisymmetric fields of third rank. Note that we also used the Bessel-Hagen method for the linearized Gauss-Bonnet gravity model in [14]. In this article, we follow the notation of Bessel-Hagen, which differs from the previous article in that we refer to coordinate transformation δx^τ as $\triangle x^\tau$.

2.2.2 Kalb-Ramond Field

We start our application of the Bessel-Hagen method with the Kalb-Ramond field: a second rank totally antisymmetric field model. Historically totally antisymmetric field models (Abelian generalizations) were introduced by N. Kemmer in 1938 [118] before generalization of electrodynamics to non-Abelian, non-linear, Yang-Mills theory was even published [196]. The gauge invariance of these models was mentioned first by Ogievetsky and Polubarinov [162] and only later such models became a popular subject of research; antisymmetric tensor fields are widely used in string models [115, 54, 157] and as well as in some supersymmetric models [53, 52, 180].

The action of electrodynamics can be viewed as the first in a series of such (similar) models with contraction of independently gauge invariant "field strength" tensors, the next one (which we refer to as Kalb-Ramond theory) being the Abelian gauge theory of a second rank antisymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$ with a vector gauge parameter θ_ν,

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu\theta_\nu - \partial_\nu\theta_\mu, \tag{2.38}$$

and the third rank field strength,

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu}, \tag{2.39}$$

which is invariant under transformation (2.38),

$$\delta_\theta F_{\mu\nu\rho} = 0, \tag{2.40}$$

and totally antisymmetric in its three indices, $F_{\mu\nu\rho} = -F_{\nu\mu\rho}$, etc. The Lagrangian density is analogous to that of electrodynamics:

$$\mathcal{L}^{(2)} = -\frac{1}{6}F_{\mu\nu\rho}F^{\mu\nu\rho}. \tag{2.41}$$

The Lagrangian density (2.41) is invariant under transformations (2.38), $\delta_\theta\mathcal{L}^{(2)} = 0$, as well as the equation of motion $\delta_\theta E^{\alpha\beta} = 0$, where $E^{\alpha\beta}$ is the corresponding Euler derivative of (2.41),

$$\frac{\delta\mathcal{L}^{(2)}}{\delta A_{\alpha\beta}} = E^{\alpha\beta} = \partial_\gamma F^{\gamma\alpha\beta}. \tag{2.42}$$

To have a more complete analogy with electrodynamics, we will also introduce the Bianchi identity for this model,

$$\partial_\mu F_{\nu\rho\sigma} - \partial_\sigma F_{\mu\nu\rho} + \partial_\rho F_{\sigma\mu\nu} - \partial_\nu F_{\rho\sigma\mu} = 0. \tag{2.43}$$

Already in [118], the Lagrangian (2.41) as well as the energy-momentum tensor,

$$T_{\mu\nu} = F_{\mu\beta\gamma}F_{\nu}^{\ \beta\gamma} - \frac{1}{6}\eta_{\mu\nu}F_{\alpha\beta\gamma}F^{\alpha\beta\gamma}, \tag{2.44}$$

were given, although without discussion of gauge invariance, perhaps because the gauge revolution had not started yet, and the energy-momentum tensor was given without derivation [188] — likely just by analogy with the known energy-momentum tensor of electrodynamics. As in the case of the electrodynamic theory, the energy-momentum tensor (2.44) has a unique property: it is gauge invariant. The lack of discussion or derivation of this energy-momentum tensor is understandable, as in many cases, papers were not dedicated to conservation laws and their derivations, or because application of the Noether theorem to classical field theories was not a subject of heavy investigation in those years. Why such models are completely missing in recent review papers dedicated to Noether's theorems and conservation laws is unclear and remains to be mystery to us, especially because "canonical" conserved currents for this model can be found in literature, e.g., [117] where the "canonical" energy-momentum tensor and angular momentum were given by,

$$T^{C}_{\mu\nu} = F_{\mu\beta\gamma}A^{\beta\gamma}_{,\nu} - \frac{1}{6}\eta_{\mu\nu}F_{\alpha\beta\gamma}F^{\alpha\beta\gamma}, \tag{2.45}$$

$$M^{C}_{\alpha\mu\nu} = x_{\mu}T_{\alpha\nu} - x_{\nu}T_{\alpha\mu} + F_{\alpha\mu\sigma}A_{\nu}^{\ \sigma} - F_{\alpha\nu\sigma}A_{\mu}^{\ \sigma}, \tag{2.46}$$

where $T^{C}_{\mu\nu}$, as in the case of electrodynamic theory, is neither gauge invariant nor symmetric, unlike the original ones in [118]. In later publications the gauge invariant energy-momentum tensor (2.44) of [118] can be found, e.g. in [188] but with no discussion, derivation or comparison with a "canonical" one, published earlier [117].

The model (2.41) exhibits a very strong similarity with electrodynamic theory: gauge invariance, a Lagrangian density quadratic in "field strength" tensor, and exact gauge invariance of all equations of this model. This observation allows to simplify our analysis considerably. Identical to our calculation for electrodynamics in [15] we have the bar field variations proportional to field strength as $\bar{\delta}A_{\mu\nu}$,

$$\bar{\delta}A_{\mu\nu} = -F_{\mu\nu\tau}\triangle x^{\tau}. \tag{2.47}$$

We will also detail the Bessel-Hagen calculations for Yang-Mills theory later the article. We can now use this form of the bar field variations in the Noether identity, as we did in Section 2.1 [15]. To simplify calculations we will use the compact form of the Noether identity [159, 160], which is the Noether identity if not separated into the familiar Euler-Lagrange equation and

Noether current,

$$\bar{\delta}\mathcal{L} + \partial_\mu \left(\mathcal{L}\triangle x^\mu\right) = 0. \tag{2.48}$$

Using the Noether identity (2.48) for the group of coordinate $\triangle x^\tau$ and field (2.47) transformations, we have the Noether identity for the Kalb-Ramond fields,

$$\frac{\partial\left(-\frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\right)}{\partial\left(\partial_\rho A_{\sigma\lambda}\right)}\partial_\rho\left(\bar{\delta}A_{\sigma\lambda}\right) + \partial_\rho\left(-\frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\triangle x^\rho\right) = 0, \tag{2.49}$$

$$F^{\rho\sigma\lambda}\partial_\rho\left(F_{\sigma\lambda\tau}\triangle x^\tau\right) - \frac{1}{6}\partial_\rho\left(F_{\mu\nu\tau}F^{\mu\nu\tau}\triangle x^\rho\right) = 0.$$

Using the Bianchi identity the Noether identity (2.49) can be simplified to,

$$F^{\rho\sigma\lambda}F_{\sigma\lambda\tau}\partial_\rho\triangle x^\tau - \frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\partial_\rho\triangle x^\rho = 0. \tag{2.50}$$

Until now we have worked with a general coordinate transformation $\triangle x^\rho$. Recalling the 15 parameter conformal transformations from [15],

$$\triangle x_\alpha = a_\alpha + \omega_{\alpha\beta}x^\beta + S\,x_\alpha + 2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu, \tag{2.51}$$

we will now check that the Noether identity is satisfied for each of these parameters. For translation, $\triangle x^\rho = a^\rho$, $\partial_\rho\triangle x^\tau = 0$, and (2.50) is immediately satisfied. For rotation, $\triangle x^\rho = \omega^{\rho\lambda}x_\lambda$, and for the Noether identity (2.50) we have,

$$F^{\rho\sigma\lambda}F_{\sigma\lambda\tau}\partial_\rho\left(\omega^{\tau\lambda}x_\lambda\right) - \frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\partial_\rho\left(\omega^{\rho\sigma}x_\sigma\right) = F^{\sigma\lambda\rho}F_{\sigma\lambda\tau}\omega^\tau{}_\rho - \frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\omega^\rho{}_\rho = 0,$$

where both terms are zero: the second term due to the trace of antisymmetric $\omega^{\rho\lambda}$, and the first one is contraction of symmetric ($F^{\rho\sigma\lambda}F_{\sigma\lambda\tau}$) and antisymmetric ($\omega^{\tau\rho}$) parts. For dilatation, $\triangle x^\mu = S\,x^\mu$, and for the Noether identity (2.50) we have,

$$F^{\rho\sigma\lambda}F_{\sigma\lambda\tau}S\,\delta^\tau_\rho - \frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}\delta^\rho_\rho S = F^{\tau\sigma\lambda}F_{\sigma\lambda\tau}S - \frac{1}{6}F_{\mu\nu\tau}F^{\mu\nu\tau}DS.$$

This will work only be satisfied in $D = 6$ (6 dimensions). Similarly for conformal transformations, $\triangle x^\tau = 2\xi^\nu x_\nu x^\tau - \xi^\tau x_\nu x^\nu$, for the Noether identity (2.50) we have,

$$2F^{\tau\sigma\lambda}F_{\sigma\lambda\tau}\left(\xi^\nu x_\nu\right) - \frac{1}{6}2DF_{\mu\nu\tau}F^{\mu\nu\tau}\left(\xi^\nu x_\nu\right) = 0.$$

Conformal transfomrations also satisfy the identity only in $D = 6$. In all dimensions, we have the Poincare group satisfying the identity (10 parameters in $D = 4$, or $\frac{D(D+1)}{2}$, in general) and conservation of energy-momentum and angular momentum tensors. In $D = 6$, as is the case for electrodynamics in $D = 4$, we have conformal group invariance (with 15 parameters in $D = 4$, or $\frac{(D+1)(D+2)}{2}$, in general). Note that, like in electrodynamics, for "canonical" transformations in $D = 6$ we also have conformal invariance despite the fact that the energy-momentum tensor is neither symmetric nor gauge invariant.

To obtain conservation laws, we have to extract Euler derivatives (rewrite the Noether identity in terms of the Euler-Lagrange equation — Lagrange expressions in Noether's terminology) in (2.49), that leads to,

$$E^{\sigma\lambda}\bar{\delta}A_{\sigma\lambda} = \partial^{\rho}\left[\left(-F_{\rho}^{\ \sigma\lambda}F_{\sigma\lambda\tau} + \frac{1}{6}\eta_{\rho\tau}F_{\mu\nu\alpha}F^{\mu\nu\alpha}\right)\triangle x^{\tau}\right] = \partial^{\rho}\left[T_{\rho\tau}\triangle x^{\tau}\right], \tag{2.52}$$

where we immediately have the (correct) symmetric, gauge invariant energy-momentum tensor for Kalb-Ramond theory for the 4-parameter Poincare translation $\triangle x^{\tau} = a^{\tau}$,

$$T_{\rho\tau} = -F_{\rho}^{\ \sigma\lambda}F_{\sigma\lambda\tau} + \frac{1}{6}\eta_{\rho\tau}F_{\mu\nu\alpha}F^{\mu\nu\alpha}, \tag{2.53}$$

which is identical to the one postulated in [118]. No improvement terms or on-shell conditions are required; it is directly derived from Noether's first theorem. From (2.52), using the remaining expressions for $\triangle x^{\tau}$ from (2.51), we derive the remaining conservation laws for the angular momentum tensor,

$$M_{\rho\tau\nu} = T_{\rho\tau}x_{\nu} - T_{\rho\nu}x_{\tau}, \tag{2.54}$$

dilatation tensor,

$$D_{\rho} = T_{\rho\tau}x^{\tau}, \tag{2.55}$$

and conformal tensor,

$$C_{\rho\nu} = T_{\rho\tau}2x_{\nu}x^{\tau} - T_{\rho\nu}x_{\mu}x^{\mu}. \tag{2.56}$$

Note again that the last two (dilatation and conformal tensors) are only valid for this theory in $D = 6$.

2.2.3 Totally Antisymmetric Fields (Third Rank)

The third rank generalization for totally antisymmetric fields was also already given in Kemmer's paper [118] (see also [157], [5]). In four dimensions, we can not go further because a field strength totally antisymmetric in five indices vanishes identically, i.e., this is the last possible in $D = 4$. In higher dimensions we could consider higher ranks but this is outside of the focus of this thesis. The Lagrangian density for the totally antisymmetric fields of highest rank in $D = 4$ is,

$$\mathcal{L}^{(3)} = -\frac{1}{8} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}, \tag{2.57}$$

where the totally antisymmetric field strength tensor is,

$$F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} - \partial_\sigma A_{\mu\nu\rho} + \partial_\rho A_{\sigma\mu\nu} - \partial_\nu A_{\rho\sigma\mu}, \tag{2.58}$$

with totally antisymmetric potential $A_{\mu\nu\rho} = -A_{\nu\mu\rho} = ...$ etc. The gauge transformation is,

$$A_{\mu\nu\rho} \rightarrow A_{\mu\nu\rho} + \partial_\mu \theta_{\nu\rho} + \partial_\rho \theta_{\mu\nu} + \partial_\nu \theta_{\rho\mu}, \tag{2.59}$$

where the gauge parameter $\theta_{\mu\nu}$ is a second rank antisymmetric tensor $\theta_{\mu\nu} = -\theta_{\nu\mu}$. The field strength (2.58), Lagrangian density (2.57), and Euler-Lagrange equation,

$$E^{\alpha\beta\sigma} = \frac{\delta \mathcal{L}^{(3)}}{\delta A_{\alpha\beta\sigma}} = \partial_\gamma F^{\gamma\alpha\beta\sigma}, \tag{2.60}$$

are all independently and exactly gauge invariant (as is the case in electrodynamics and Kalb-Ramond), thus we have $\delta_\theta F_{\mu\nu\rho\sigma} = 0$, $\delta_\theta \mathcal{L} = 0$ and $\delta_\theta E^{\alpha\beta\sigma} = 0$. The Bianchi identity for this model is,

$$\partial_\mu F_{\nu\rho\sigma\gamma} + \partial_\gamma F_{\mu\nu\rho\sigma} + \partial_\sigma F_{\gamma\mu\nu\rho} + \partial_\rho F_{\sigma\gamma\mu\nu} + \partial_\nu F_{\rho\sigma\gamma\mu} = 0. \tag{2.61}$$

Again for the bar field transformations we have use the gauge freedom of the theory transformations which are proportional to the field strength tensor,

$$\bar{\delta} A_{\mu\nu\rho} = F_{\mu\nu\rho\tau} \triangle x^\tau. \tag{2.62}$$

identical to the electrodynamics calculations in [15]. Using this transformation to check the Noether identity (2.48) we are left with,

$$-F^{\rho\sigma\lambda\gamma} \partial_\rho \left(F_{\sigma\lambda\gamma\tau} \triangle x^\tau \right) + \partial_\rho \left(-\frac{1}{8} F_{\mu\nu\lambda\sigma} F^{\mu\nu\lambda\sigma} \triangle x^\rho \right) = 0.$$

Performing exactly the same calculations as in the case of electrodynamics and Kalb-Ramond we have gauge invariant conservation laws for the translation and rotation as well as for full conformal group in a specific dimension ($D = 8$ in the case of this model). These are identical to (2.53) for energy-momentum, (2.54) for angular momentum, (2.55) for dilatation and (2.56) for conformal (instead with the field strength tensor in (2.58)).

2.2.4 Yang-Mills theory

We will now consider the prototypical non-Abelian gauge theory, Yang-Mills theory [196], to show that the Bessel-Hagen method is not restricted to the Abelian gauge theories we have considered so far. The Lagrangian density of the Yang-Mills action is,

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{2.63}$$

where $F_{a\mu\nu}$ is the field strength tensor,

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + C_{abc}A^b_\mu A^c_\nu. \tag{2.64}$$

Here C_{abc} is the totally antisymmetric structure constant ($C_{abc} = -C_{bac}$, etc.). Greek indices $\mu, \nu, ...$ refer to space-time ($\mu = 0, 1, 2, 3$), whereas Latin indices are internal, e.g., $F^a_{\mu\nu}$ is a covariant second rank antisymmetric tensor. Index a ($a = 1, 2, 3$) in A^a_ν and $F^a_{\mu\nu}$ counts fields and their field strengths (we follow convention of Weinberg [191]). The gauge transformation of A^a_μ is,

$$\delta_\theta A_{a\mu} = \partial_\mu\theta_a + C_{abc}A^b_\mu\theta^c \equiv D_\mu\theta_a, \tag{2.65}$$

which is the gauge transformation of Yang-Mills theory (θ_a are arbitrary real functions of space-time coordinates, D_μ is a so-called"covariant derivative"). The Euler-Lagrange equation (equations of motion) for the theory is,

$$E^\mu_a = \frac{\delta\mathcal{L}_{YM}}{\delta A^a_\mu} = \frac{\partial\mathcal{L}_{YM}}{\partial A^a_\mu} - \partial_\nu\frac{\partial\mathcal{L}_{YM}}{\partial\left(\partial_\nu A^a_\mu\right)} = C_{abc}A^b_\nu F^{c\nu\mu} + \partial_\nu F^{\nu\mu}_a = D_\nu F^{\nu\mu}_a. \tag{2.66}$$

Neither the field strength nor Lagrange expressions for the Yang-Mills theory are exactly gauge invariant (they are proportional to themselves $\delta_\theta F_{a\mu\nu} = C_{abc}F^b_{\mu\nu}\theta^c$ and $\delta_\theta E^\mu_a = C_{abc}E^{b\mu}\theta^c$). However, because of these relations and antisymmetry of the structure constant, the Lagrangian density is exactly gauge invariant,

$$\delta_\theta F^a_{\mu\nu}F^{\mu\nu}_a = F^{\mu\nu}_a\delta_\theta F_{a\mu\nu} = C_{abc}F^{\mu\nu}_a F^b_{\mu\nu}\theta^c = 0,\ \delta_\theta\mathcal{L}_{YM} = 0,$$

and also quadratic in field strength that makes it very similar with the other theories we considered, despite of non-linearity of the Yang-Mills Lagrangian (2.64) and equations of motion (2.66). The exactly gauge invariant Lagrangian density allows for straightforward application of the Bessel-Hagen method — for application of Noether's first theorem, invariance of an action is a nessesary condition as well as its invariance under gauge transformations that was used in the Bessel-Hagen derivation, i.e., we have both these conditions in Yang-Mills theory. From the treatment of previous models it is clear that the use of the Bianchi identities can considerably simplify calculations. For the Yang-Mills field strength we have the identity,

$$D_\mu F^a_{\nu\lambda} + D_\lambda F^a_{\mu\nu} + D_\nu F^a_{\lambda\mu} = 0, \tag{2.67}$$

As in the previous cases the bar transformation of fields is identically,

$$\bar{\delta} A^a_\mu = F^a_{\mu\nu} \triangle x^\nu, \tag{2.68}$$

which we will calculate in detail at the end of this section. In fact, transformation (2.68) in the case of the Yang-Mills theory is not new and was considered by Jackiw [110], although his presentation was very ad-hoc and connection to Noether's first theorem was not explicitly demonstrated or even discussed, but as the origin of such modification, the author referred to previous investigations of (extended) supersymmetric models; this so-called "the gauge-covariant translation"[60], naturally arises in the composition law for two infinitesimal sypersymmetry transformations that involve infinitesimal space-time (coordinate) transformations. Note that (2.68) here is not just a translation, and $\triangle x^\nu$ are all infinitesimal transformations of conformal group. The authors of [60] noticed that the Noether procedure for a covariant translation leads to a gauge-invariant expression for the energy-momentum tensor, but following the convention in the literature, turned in their discussion back to the "canonical" energy-momentum tensor and corresponding Belinfante improvement.

The short ad-hoc presentation of [110] is probably the reason that this article, as well as the proper transformations for Yang-Mills (2.68), are usually not even mentioned in review papers on conservation laws; even in the recent review [29], where an almost complete list of works dedicated to direct application of Noether's theorem was given (see [37-40,50] of [29]), the paper of Jackiw (as well as Bessel-Hagen) are missing.

The most basic Noether identity for Yang-Mills includes self interaction terms thus we have,

$$\frac{\partial \left(\mathcal{L}_{YM}\right)}{\partial \left(A_{a\mu}\right)} \left(\bar{\delta} A_{a\mu}\right) + \frac{\partial \left(\mathcal{L}_{YM}\right)}{\partial \left(\partial_\mu A_{a\lambda}\right)} \partial_\mu \left(\bar{\delta} A_{a\lambda}\right) + \partial_\mu \left(\mathcal{L}_{YM} \triangle x^\mu\right) = 0. \tag{2.69}$$

Extracting the Euler derivatives (2.66) and rearranging (2.69) becomes,

$$E^\mu_a \bar{\delta} A^a_\mu + \partial_\mu \left(\frac{\partial \mathcal{L}_{YM}}{\partial \left(\partial_\mu A^a_\lambda\right)} \bar{\delta} A^a_\lambda + \mathcal{L}_{YM} \triangle x^\mu \right) = 0. \tag{2.70}$$

Using (2.63), (2.66), and (2.68) on (2.70) we have,

$$\left(\partial_\rho F_a^{\rho\mu} + C_{abc}A_\rho^b F^{c\rho\mu}\right) F_{\mu\nu}^a \triangle x^\nu + \partial_\mu \left(-F^{a\mu\lambda}F_{\lambda\nu}^a \triangle x^\nu - \frac{1}{4}F_{\lambda\nu}^a F_a^{\lambda\nu}\triangle x^\mu\right) = 0. \tag{2.71}$$

Substituting each of the conformal transformations (2.51) into (2.71) we find that the identity is satisfied for all Poincare group parameters for all dimensions, and for the full 15 parameter conformal from in $D = 4$. Gauge invariance is explicit by rearranging (2.71),

$$E_a^\mu F_{\mu\nu}^a \triangle x^\nu = \partial_\mu \left(\left(F^{a\mu\lambda}F_{\lambda\nu}^a + \frac{1}{4}\eta_{\mu\nu}F_{\lambda\rho}^a F_a^{\lambda\rho}\right)\triangle x^\nu\right). \tag{2.72}$$

The Lagrangian density is gauge invariant as well as the first term of (2.72)

$$\delta_\theta \left(F^{a\mu\lambda}F_{\lambda\nu}^a\right) = \delta_\theta \left(F^{a\mu\lambda}\right) F_{\lambda\nu}^a + F^{a\mu\lambda}\delta_\theta F_{\lambda\nu}^a = C_{abc}\left(F^{b\mu\lambda}F_{\lambda\nu}^a + F^{a\mu\lambda}F_{\lambda\nu}^b\right)\theta^c = 0$$

This form is completely similar to electrodynamics, i.e. there is a gauge invariant, symmetric and traceless energy-momentum tensor $T^{\mu\nu}$ from the 4-parameter Poincare translation $\triangle x^\nu = a^\nu$,

$$T^{\mu\nu} = F^{a\mu\lambda}F_{\lambda}^{a\ \nu} + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\rho}^a F_a^{\lambda\rho}. \tag{2.73}$$

In $D = 4$, all Noether currents immediately follow for the specific values for $\triangle x^\nu$ (of the 15-parameter conformal group), similar with Abelian models where e.g. (2.54) was the angular momentum tensor, (2.55) was the dilatation tensor and (2.56) was the conformal tensor, each being in gauge invariant form built from the Yang-Mills field strength tensor (2.64).

If we were to simply use the conformal transformations, the Noether identity is satisfied; we have the 15-parameters group of transformations and all currents, despite of the non-symmetric, non-traceless, and non-gauge invariant energy-momentum tensor that follows. All of the expressions due to Noether's first theorem. The words of Bessel-Hagen are perfectly applied to the canonical expressions for currents, that they "*...are very long and complicated. But the main lack of the relations is that they contain the components of the four-potential explicitly, and not only via the combinations having physical significance*", or simply, as Jackiw wrote [110], because "*...the form of conformal currents is rather inelegant*". It would be difficult to disagree with such a characterization.

Application of the Bessel-Hagen method to Yang-Mills theory

Using the Bessel-Hagen method we can determine gauge parameters that, together with canonical transformations, lead to transformation (2.68). This was completed for electrodynamics in [15]. For completeness, we briefly describe application of the Bessel-Hagen procedure to the

Yang-Mills action. As in the case of the Maxwell theory, we can consider gauge transformations as a special case of Noether's theorem with only transformation of a potential, which is a gauge transformation with unspecified scalar functions,

$$\Delta A_{a\mu} = \delta_\theta A_{a\mu} = \partial_\mu \theta_a + C_{abc} A^b_\mu \theta^c. \tag{2.74}$$

In such a case, with only transformations of fields, Noether's identity (2.70) becomes,

$$E_{a\nu} \Delta A^\nu_a = \partial_\mu \left(-\frac{\partial \mathcal{L}_{YM}}{\partial \left(\partial_\mu A_{a\nu} \right)} \Delta A^\nu_a \right). \tag{2.75}$$

The superposition of two (gauge and canonical identities) gives,

$$E_{a\mu} \bar{\delta} A^{a\mu} = \partial_\mu \left(-F^{a\mu\rho} \bar{\delta} A^{a\mu} - \eta_{\nu\mu} \mathcal{L}_{YM} \triangle x^\nu \right), \tag{2.76}$$

where,

$$\bar{\delta} A^{a\mu} = -\partial_\rho \left(\triangle x_\tau \right) A^\tau_a + \partial_\rho \theta_a + C_{abc} A^b_\rho \theta^c - \partial_\tau A_{a\rho} \triangle x^\tau. \tag{2.77}$$

For any value of gauge parameters θ_a we will have Noether's identity and conservation laws with different Noether's currents; any scalar constructed from $\triangle x_\tau$ and potentials will satisfy the identity. Of course, the simplest scalar is $\theta_a = A^\tau_a \triangle x_\tau$, but any other will also satisfy the identity, e.g., $\theta_a = \partial^\mu A^\tau_a \partial_\mu \triangle x_\tau$ or $\theta_a = A^\rho_b A^b_\rho A^\tau_a \triangle x_\tau$, etc. Using the Bessel-Hagen method to determine a gauge invariant energy-momentum tensor (Noether current) we can set,

$$-\partial_\rho \left(\triangle x_\tau \right) A^\tau_a + \partial_\rho \theta_a + C_{abc} A^b_\rho \theta^c - \partial_\tau A_{a\rho} \triangle x^\tau = F_{a\rho\tau} \triangle x^\tau \tag{2.78}$$

and solve for θ_a,

$$\theta_a = A^\tau_a \triangle x_\tau, \tag{2.79}$$

that together with canonical transformations (2.77) gives our (2.68). Among many mathematically correct conservation laws that can be found using Noether's approach for the Yang-Mills theory, there is one that has physically important property of being gauge invariant and corresponds to one (among many) very specific transformations that preserves invariance of the Yang-Mills Lagrangian density. The same can be said for all of the models treated in this article.

2.2.5 Conclusions

The contemporary controversy surrounding the need to improve the "canonical" Noether energy-momentum tensor is one which was solved by a contemporary of Noether, Erich Bessel-Hagen, with her participation, in 1921. Decades later, with no English translation of this article and other possible historical reasons outlined in this thesis, Bessel-Hagen's work on deriving physical conservation laws from Noether's first theorem was largely forgotten. A problem arose as the next generation attempted to apply Noether's methods to problems in relativistic field theory; the resulting "canonical" Noether energy-momentum tensor was not what was expected for known theories. Countless improvements were proposed to provide a temporary ad-hoc fix to the problem and which could allow one to obtain well known energy-momentum expressions. In this article show that Bessel-Hagen's procedure can be applied to several models found in the literature beyond electrodynamics. Despite existence of a few later rediscoveries of his result (e.g. [70, 153, 155, 186]), the Bessel-Hagen approach, in our opinion, remains to be superior, in its clarity and generality, demonstration of direct application of the original Noether formulation: providing a very uniform derivation for all conservation laws (not just an energy-momentum tensor as it is often discussed); explicitly connecting both finite (coordinates) and infinite (gauge) continuous symmetry groups and demonstrating that all steps of derivation are just special cases of the original and very general Noether results. We also pointed out that to choose conservation laws from a variety of possible mathematically correct conservation laws, additional (physical) arguments should be involved (e.g. gauge invariance) that for the considered in this paper examples allow one to determine unique expressions.

We would like to emphasize that the controversy surrounding the "improvement" of Noether's first theorem that has existed throughout the literature for some time had absolutely nothing to do with Noether or any deficiency in her theorems; she made no attempt to apply her theorems to specific models in her article, and any controversy that existed was created externally to her analytic approach. The "canonical Noether" energy-momentum tensor appears nowhere in her article and she made no such contribution; the naming of the "canonical Noether" energy-momentum tensor should more appropriately be referred to as the "canonical" energy-momentum tensor to avoid implying that Noether was in some way responsible for this problematic non-physical expression.

There is a much more important question, which is directly related to the Noether and Bessel-Hagen papers, the question about current treatment of conservation laws in physics, and in particular, the question about equivalence of different approaches accepted and used in the physics literature and Noether's results. There are numerous different methods for deriving an energy-momentum tensor which raises many ambiguity and non-uniqueness questions. These questions will be the focus of the next chapter (Chapter 3) of this thesis.

It would be more appropriate to rightfully use Noether's name as "Noetherian currents" with a very specific meaning that should be clearly stated and easily understood: currents that follow from Noether' first theorem. There are standard terms (in Mathematics) with Noether's name attached, e.g., "(non)-Noetherian rings" which is related to the main area of her research (Algebra). In physics, we have a different situation: all relativistic classical field theory models have actions (variational problems), and in this sense they are all subjects of the Noether theorems. This is not enough because of the existence of the variety of methods (for e.g. energy-momentum tensor derivation) that do not have direct (if any) connection to the Noether variational problem. Although in some simple cases they give the same results with Noether's first theorem, there is no general proof of their equivalence. Moreover, there are strong indications that there is no equivalence, as we discuss in Chapter 3 [13]. Currents obtained by such methods can be defined as Noetherian or non-Noetherian if they satisfy the Noether identity for some choice of variational symmetries or if such symmetries cannot be found.

In our opinion, if we have some currents which are non-Noetherian, they should be rejected, and before we use any alternative method, its equivalence with Noether's theorems must be proven in general (proofs for some very restricted cases is not enough to justify the frequent general claims in the literature). However, if an opposite view prevails and physicists continue to use (and accept) different results obtained by different methods that produce non-Noetherian currents, then Noether's theorems should not be called a "fundamental, great, important, major, etc." result by those who are using methods that contradict her methodology. The acceptance of non-Noetherian currents for relativistic field theories and statements about importance of Noether's results are simply not compatible.

Chapter 3

Towards uniqueness of the energy-momentum tensor

This chapter focuses on the problem outlined in Section 1.5.2 of the Introduction; the ambiguity in multiple contradictory methods for deriving the conservation laws (and in particular, energy-momentum tensors) in Minkowski spacetime. Several methods exist, and due to the coinciding results for simple models, they are all asserted to be ”energy-momentum” tensors. This chapter includes 3 articles which address this problem [13, 10, 8]. The first article [13] in Section 3.1 disproves the notion of general equivalence of the two most common methods for deriving an energy-momentum tensor: the Noether and Hilbert methods, which was published in *Nuclear Physics B*. The second article [10] in Section 3.2 explores the more general class of energy-momentum tensors for the general linear system of Klein-Gordon Lagrangian densities, proving several results related to the relationship of these expressions, which shows that divergence of the definitions is possible even for the case of a scalar field theory. Finally [8] in Section 3.3 the non-uniqueness problem of energy-momentum tensors in linearized gravity is address and several results are presented, most notably that there are infinitely many expressions that can be obtained from the conventional superpotential “improvement” method of the canonical Noether energy-momentum tensor. This article has been published in the journal *Classical and Quantum Gravity*. Together these articles highlight the problem of having numerous distinct mathematical definitions for something which is supposedly the same physical energy-momentum tensor $T^{\mu\nu}$. Only the first article [13] in Section 3.1 is co-authored, and details of the contributions are left to the Co-Authorship Statement.

3.1 Noether and Hilbert (metric) energy-momentum tensors are not, in general, equivalent

Abstract Multiple methods for deriving the energy-momentum tensor for a physical theory exist in the literature. The most common methods are to use Noether's first theorem with the 4-parameter Poincaré translation, or to write the action in a curved spacetime and perform variation with respect to the metric tensor, then return to a Minkowski spacetime. These are referred to as the Noether and Hilbert (metric/ curved space/ variational) energy-momentum tensors, respectively. In electrodynamics and other simple models, these two methods yield the same result. Due to this fact, it is often asserted that these methods are generally equivalent for any theory considered, and that this gives physicists a freedom in using either method to derive an energy-momentum tensor depending on the problem at hand. This ambiguity in selecting one of these two different methods has gained attention in the literature, but the only attempted proofs of general equivalence of the two methods are for at most first order derivatives of first rank (vector) field theories. For spin-2, the ideal candidate to check this equivalence for a more complicated model, there exist many energy-momentum tensors in the literature, none of which are gauge invariant, so it is not clear which expression one hopes to obtain from the Noether and Hilbert approaches unlike in the case of e.g. electrodynamics. It has been shown, however, that the linearized Gauss-Bonnet gravity model (second order derivatives, second rank tensor potential) has an energy-momentum tensor that is unique, gauge invariant, symmetric, conserved, and trace-free when derived from Noether's first theorem (all the same properties of the physical energy-momentum tensor of electrodynamics). This makes it the ideal candidate to check if the Noether and Hilbert methods coincide for a more complicated model. It is proven here using this model as a counterexample, by direct calculation, that the Noether and Hilbert energy-momentum tensors are not, in general, equivalent.

3.1.1 Motivation

The energy-momentum tensor of a physical field theory is an expression of fundamental significance to a physical model. In electrodynamics, for example, it compactly expresses familiar conservation laws and the Lorentz force law upon differentiation. However, the procedure to derive an energy-momentum tensor from a Lagrangian density is not unique. Several methods for deriving this expression can be found in the literature. We will focus on the two most common procedures for deriving an energy-momentum tensor in Minkowski spacetime, the Noether and Hilbert methods [29]. Methods such as the Fock method [75, 28] will not be discussed in this article, as they don't involve a procedure to derive the energy-momentum tensor from a Lagrangian density.

For a recent summary of the Noether and Hilbert methods, we will refer the reader to the paper by Blaschke, Gieres, Reboud and Schweda 'The energy-momentum tensor(s) in classical gauge theories' published in Nuclear Physics B in 2016 [29]. We will refer to the paper as BGRS [29] due to the frequent reference to their paper our article. The BGRS paper has an extensive summary of the literature, so we suggest turning to BGRS [29] and the references therein if the reader is unfamiliar with these topics. It is well known that for a vector field (electrodynamics) and for a scalar field (Klein-Gordon), the Noether and Hilbert methods coincide with the same energy-momentum tensor. In BGRS [29] the authors address this question for Yang-Mills and spinor fields and conclude again that these are equivalent from both Noether and Hilbert approaches, yet again those are models with at most first order derivatives of a vector potential in the action. Any attempted proof of equivalence of the Noether and Hilbert methods has been limited to simple models with at most first order derivatives of a vector potential in the action [32, 176, 76, 133, 170]. Unfortunately none of these authors considered a more complicated model to test the equivalence, as it only takes one counterexample to disprove the notion of general equivalence; this is what will be provided in the present article.

In this article we will focus on the very specific question: for more complicated models in Minkowski spacetime, do the Noether and Hilbert methods yield an equivalent result? In other words, do actions with higher order derivatives and higher ranks of tensor potential, such as the linearized higher derivative gravity models, yield the same energy-momentum tensor by following the Noether and Hilbert procedures. The ideal candidate to explore this question, spin-2, is problematic because it has been proven that there exists no gauge invariant energy-momentum tensor for that model [140], and there is no generally accepted unique energy-momentum tensor for the theory, as many exist in the literature [163, 28]. This issue has come to the forefront recently regarding the necessity to have a well defined energy-momentum tensor for the spin-2 field to self couple in the standard spin-2 to general relativity derivations [163, 41, 63, 28]. Therefore it is not clear which expression one hopes to obtain from both the Noether and Hilbert method for spin-2 as in the case of electrodynamics where a single, accepted physical tensor exists.

A more complicated ideal candidate does exist, in the form of the linearized higher derivative gravity models, that is the models built from the contracted linearized Riemann tensor $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}R^{\mu\nu}$ and Ricci scalar R^2. These relativistic models in Minkowski spacetime have second order derivatives of a second rank symmetric tensor potential $h_{\mu\nu}$ in the action (terms of the form $\partial\partial h \partial\partial h$). In particular, we will consider the energy-momentum tensor for the linearized Gauss-Bonnet gravity model, which has been well known to string theorists and other researchers for some time [156]. This expression has been shown to be derived from Noether's first theorem [14], and it is unique, gauge invariant, symmetric, conserved, and trace-free; all the properties of a physical energy-momentum tensor as defined by BGRS [29]. It is these properties of electrodynamics that allows the equivalence between the Noether and Hilbert methods to be accurately concluded. Using this example of the linearized higher derivative gravity models, and in particular the linearized Gauss-Bonnet gravity model, we give a proof by counterexample that the Noether and Hilbert energy-momentum tensors are not, in general, equivalent. We then outline why this result will hold for higher order of derivative/ higher rank of tensor potential models more generally. If several methods exist, and they do not generally yield the same result, it is an issue of fundamental significance as to which method is truly allowing one to derive physical results for any general action, and which happen to coincide for actions of simple physical models.

3.1.2 The Noether and Hilbert methods for deriving energy-momentum tensors in Minkowski spacetime

From Noether's theorem the energy-momentum tensor for electrodynamics was directly derived by Bessel-Hagen in 1921 [26], without the need for improvements, by considering the gauge symmetry of the action. Several other authors came to a similar conclusion later [39, 153, 70, 155], apparently unaware of Bessel-Hagen's paper, which was only recently translated into English [108]. If the action is exactly gauge invariant, this procedure derives the physical energy-momentum tensor for the theory without the need to add any ad-hoc improvement terms to obtain a gauge invariant expression. In BGRS [29], the authors outline this procedure in section 2.2.2, but without referring to Bessel-Hagen, only to [153, 70, 155]. We will refer to this as the Bessel-Hagen method because he was the first to present this procedure, and in our opinion, in the clearest and most direct way based on Noether's original work.

It is important to briefly mention the 'improvement' of energy-momentum tensors derived from Noether's theorem in the literature, due to its widespread use. Various improvements exist and are well summarized in BGRS [29]. Conventional wisdom states that one can improve Noether's energy-momentum tensor when the result one obtains from Noether's first theorem is not the physically accepted expression for the energy-momentum tensor. This involves adding terms which do not follow from Noether's theorem in order to obtain the desired result. Since the Bessel-Hagen method derives the correct, physical expressions directly from Noether's theorem without the need to add any terms, it is not really an 'improvement' (no ad-hoc terms need to be added), rather it is the correct derivation intended by Noether, who is cited as giving Bessel-Hagen the ideas for his paper. We note that the most common improvement found in the literature is the Belinfante method [23], designed to build a symmetric energy-momentum tensor from the non-physical 'canonical Noether' energy-momentum tensor. This improvement does not guarantee gauge invariance of the energy-momentum tensor, a deficiency addressed by a new improvement procedure of BGRS [29].

This sentiment was summarized in BGRS [29], namely the importance of a gauge invariant energy-momentum tensor for theories considered physical, and the deficiencies of the Belinfante method [23]: *'If one considers gauge field theories in Minkowski space as we do in the present article, then the EMT* [energy-momentum tensor] *necessarily has to be gauge invariant due to its physical interpretation. However, Belinfante's improvement procedure does not yield a priori a gauge invariant EMT when applied to gauge theories, and in addition it does not work in the straightforward manner for the physically interesting case where matter fields are minimally coupled to a gauge field.'* In the cases of electrodynamics and linearized Gauss-Bonnet gravity, the accepted physical, unique, gauge invariant, symmetric, conserved, and trace-free expressions are obtained from the Bessel-Hagen method, so there is no need to add improvement terms to the Noether result for these models.

Noether's first theorem is used to derive conservation laws by considering the action $S = \int \mathcal{L} dx$ to be invariant under simultaneous variation of the coordinates δx_ν and fields $\delta\Phi_A$ (where $\mathcal{L}(\Phi_A, \partial_\mu\Phi_A, \partial_\mu\partial_\nu\Phi_A, \dots)$ is the Lagrangian density, A represents any rank of tensor potential Φ_A and $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is abbreviated notation for a derivative). From Noether's first theorem we have the relationship between the Euler-Lagrange equation and some total derivative [159, 124, 88],

$$\begin{aligned}&\left(\frac{\partial\mathcal{L}}{\partial\Phi_A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)} + \partial_\mu\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)} + \dots\right)\delta\Phi_A \\ &+ \partial_\mu\left(\eta^{\mu\nu}\mathcal{L}\delta x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\partial_\omega\delta\Phi_A - \left[\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\right]\delta\Phi_A + ...\right) = 0\,. \quad (3.1)\end{aligned}$$

Using Equation (4.20) and the Bessel-Hagen method we can derive the standard energy-momentum tensor $T_N^{\mu\nu} = F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ for electrodynamics (with the field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$) from Noether's theorem by use of the 4-parameter Poincaré translation and gauge invariance of the action, where subscript N will be used to identify any physical expression derived from Noether's theorem.

The other most common procedure for deriving an energy-momentum tensor in Minkowski spacetime is the Hilbert method, sometimes referred to as the metric energy-momentum tensor, curved space energy-momentum tensor, and even variational energy-momentum tensor. A good summary of this method is found in BGRS [29], Section 3 'Einstein-Hilbert EMT in Minkowski space'. The authors refer to this tensor as the metric energy-momentum tensor in their article. The Hilbert energy-momentum tensor is derived by writing the action in 'curved space' by replacing all ordinary derivatives with covariant derivatives $\partial \to \nabla$, replacing the Minkowski metric with the general metric tensor $\eta \to g$, and inserting the Jacobian term $\sqrt{-g}$. After expressing the action in this form, the variation with respect to the general metric tensor is performed,

$$\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}} = \frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} - \partial_\omega \frac{\partial\mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} + \partial_\xi\partial_\omega \frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} + \ldots \quad . \tag{3.2}$$

Once the variational derivative is found from this procedure, it is then 'returned to flat space' by replacing the metric tensors with the Minkowski metric, yielding an energy-momentum tensor of the form,

$$T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}} \frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}\bigg|_{g=\eta} . \tag{3.3}$$

Note that this definition is given in equation (3.18) in BGRS [29], where we take the + instead of − expression here so that the signs match the derivation for electrodynamics in the following section (both signs can be found throughout the literature depending on convention). The subscript H will indicate what is derived from the Hilbert method. Remarkably, for electrodynamics, these two expressions coincide $T_N^{\gamma\rho} = T_H^{\gamma\rho}$. Due to this coincidence, and the fact that electrodynamics has a unique physical energy-momentum tensor accepted in the literature, it is tempting to assert general statements about their equivalence. Other simple models amplify these sentiments, leading to the belief that the results of these methods are in some sense generally equivalent. However, they have only been reconciled for simple scalar or vector fields and first order derivatives in actions. Higher order of derivative, higher rank of tensor potential models, such as those presented in this article, have not previously been considered to verify the general equivalence of the Noether and Hilbert methods.

In BGRS [29], the authors remark about the Noether tensor $T_N^{\gamma\rho}$ (including improvements) vs. the Hilbert tensor $T_H^{\gamma\rho}$ by stating *'This definition of the EMT in Minkowski space is conceptually and mathematically quite different from the one of* $T^{\mu\nu}_{imp}[\psi]$ *which we presented in section 2 and which follows from Noether's theorem (eventually supplemented by an improvement procedure to render the canonical expression of the EMT symmetric in its indices or gauge invariant, or both symmetric and traceless).'* They go on to consider at most first order, vector models as is common in the literature *'In the following, we will show that the two definitions for the EMT's of YM-theories in Minkowski space,* [...]*which results from the coupling to gravity, and the improved EMT* [...]*which follows from Noether's first theorem supplemented by the "gauge improvement" procedure, coincide with each other.'*. We note that the Bessel-Hagen method can be used to derive the physical energy-momentum tensor for electrodynamics and Yang-Mills theory directly from Noether's theorem without the need for any improvements.

This is why a higher derivative model, such as a linearized higher derivative gravity model, is so important to consider. To explore the question of general equivalence between the Noether and Hilbert methods, we must check if they coincide beyond simple physical models that we already know.

3.1.3 Equivalence of the Noether and Hilbert expressions for classical electrodynamics

Before comparing the Noether $T_N^{\gamma\rho}$ and Hilbert $T_H^{\gamma\rho}$ for the linearized Gauss-Bonnet model, it is best to recap the equivalence $T_N^{\gamma\rho} = T_H^{\gamma\rho}$ for electrodynamics, to show how to perform these derivations for a simple model before moving on to the higher order case. The physical energy-momentum tensor for electrodynamics, which was known before the publication of Noether's theorems, was first derived by Bessel-Hagen in 1921 [26]. Using equation (4.20), he derived this expression directly from the standard electromagnetic Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ for the field strength tensor $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. For a theory with a Lagrangian built from terms quadratic in first order derivatives of a vector potential, Equation (4.20) simplifies to,

$$\left(\partial_\gamma \frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\nu)}\right)\delta A_\nu = \partial_\gamma\left(\eta^{\gamma\nu}\mathcal{L}\delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\nu)}\delta A_\nu\right), \tag{3.4}$$

where the equation of motion (left hand side) forms an identity with the conservation law (right hand side). For a conformally invariant theory such as electrodynamics the 15 parameter conformal group of transformations that will leave the action invariant is $\delta x_\alpha = a_\alpha + \omega_{\alpha\beta}x^\beta + S x_\alpha + 2\xi_\nu x_\alpha x^\nu - \xi_\alpha x_\nu x^\nu$. The first term, the 4 parameter translation of the Poincaré group, is the 'symmetry' that corresponds to energy-momentum tensors derived from Noether's theorem. The transformation of fields δA_ν that leave the action invariant are defined generally for Noether's first theorem as $\delta A_\nu = \delta A'_\nu - \partial^\beta A_\nu \delta x_\beta$ [159, 124, 26, 88]. The first term, $\delta A'_\nu$, is related to field transformations that leave the action invariant (ie the spin-1 gauge transformation); this was neither discussed nor specified by Noether and could be anything (i.e., gauge symmetries) that preserves invariance of the action. Bessel-Hagen showed that using gauge invariance of the action to define $\delta A'_\nu$ that the transformation of the potential is exactly $\delta A_\nu = F_{\nu\rho}\delta x^\rho$. Inserting this, $\delta x_\rho = a_\rho$ and $\frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\nu)} = -F^{\gamma\nu}$ into Equation (3.4) we have,

$$\left(-\partial_\gamma F^{\gamma\nu}\right)\delta A_\nu = a_\rho \partial_\gamma\left(F^{\gamma\nu}F^\rho{}_\nu - \frac{1}{4}\eta^{\gamma\rho}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{3.5}$$

Therefore the energy-momentum tensor for electrodynamic theory $T_N^{\gamma\rho} = F^{\gamma\nu}F^\rho{}_\nu - \frac{1}{4}\eta^{\gamma\rho}F_{\alpha\beta}F^{\alpha\beta}$ is derived directly from Noether's first theorem.

The Hilbert energy-momentum tensor for electrodynamics is derived by equation (3.49) after expressing the standard Lagrangian in curved space form, namely replacing the Minkowski metrics with the metric tensor, replacing all ordinary derivatives with covariant derivatives $\partial \to \nabla$, replacing all Minkowski metrics with general metrics $\eta \to g$, and inserting the Jacobian $\sqrt{-g}$. Starting by re-writing the field strength tensor in terms of covariant derivatives $F^{\nabla}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha - \partial_\nu A_\mu + \Gamma^\alpha_{\nu\mu} A_\alpha = F_{\mu\nu}$, we see that the Γ part exactly cancels itself, recovering the original field strength tensor. It should be noticed that for higher derivative models of higher rank potentials, many extra Γ parts remain without cancellation, creating many more terms in the final energy-momentum tensor. This is likely part of the reason why for simple models the two methods $T^{\gamma\rho}_N = T^{\gamma\rho}_H$ coincide. Therefore for the curved space Lagrangian density we have,

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{3.6}$$

This simplifies the Euler derivative to just including derivatives of the metric, leaving for the Hilbert energy-momentum tensor $T^{\gamma\rho}_H = \frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}}|_{g=\eta}$. Taking the derivative with respect to the metric, we use $\frac{\partial\sqrt{-g}}{\partial g_{\gamma\rho}} = \frac{1}{2}g^{\gamma\rho}\sqrt{-g}$ and $\frac{\partial g^{\lambda\nu}}{\partial g_{\beta\gamma}} = -\frac{1}{2}(g^{\beta\lambda}g^{\gamma\nu} + g^{\gamma\lambda}g^{\beta\nu})$. Performing this differentiation we have $\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} = \frac{1}{2}\sqrt{-g}F_{\alpha\beta}F_{\mu\nu}(g^{\nu\beta}g^{\rho\mu}g^{\gamma\alpha} - \frac{1}{4}g^{\gamma\rho}g^{\beta\nu}g^{\alpha\mu})$. Therefore $T^{\gamma\rho}_H = \frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}}|_{g=\eta} = F^{\gamma\nu}F^{\rho}{}_{\nu} - \frac{1}{4}\eta^{\gamma\rho}F_{\alpha\beta}F^{\alpha\beta}$ which exactly coincides with what is derived from the Noether method, $T^{\gamma\rho}_N = T^{\gamma\rho}_H$.

The fact that the two derivations yield the same result is of fundamental interest since the two methods are mathematically quite different, as the authors of BGRS [29] noted. The problem is, these two expressions are only ever calculated for simple models with first order derivatives of at most a vector potential in the action. Both attempts at a general proof [76, 170] also rely on these simple models. We will now consider the linearized Gauss-Bonnet model which has a physical, unique, symmetric, gauge invariant, conserved and trace-free energy-momentum tensor derived using Noether's first theorem, as in the case of electrodynamics. As we will see, this greatly complicates the Hilbert expression due to second order derivatives and second rank tensor potential of the model, yielding a proof by counterexample that the Noether and Hilbert energy-momentum tensors are not generally equivalent.

3.1.4 Non-Equivalence of the Noether and Hilbert expressions for Linearized Gauss-Bonnet gravity

We will now consider the linearized Gauss-Bonnet gravity model (a relativistic model in Minkowski spacetime) that has a well known energy-momentum tensor derived from the Noether method. Here we will derive the Hilbert (metric) energy-momentum tensor in Minkowski spacetime as outlined by BGRS [29], and compare to the Noether result to see if they are truly equivalent for this more complicated model. The Lagrangian density for this model is

$$\mathcal{L} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \tag{3.7}$$

where the scalars are built from contraction of the linearized Riemann tensor $R_{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$ and Ricci scalar R:

$$R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha}), \tag{3.8}$$

$$R^{\nu\beta} = \eta_{\mu\alpha}R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\beta\partial^\alpha h^\nu_\alpha + \partial^\nu\partial^\alpha h^\beta_\alpha - \Box h^{\nu\beta} - \partial^\nu\partial^\beta h), \tag{3.9}$$

$$R = \eta_{\nu\beta}R^{\nu\beta} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \tag{3.10}$$

Each of these R's will indicate these linearized expressions unless otherwise noted. The energy-momentum tensor for Gauss-Bonnet gravity has been well known to string theorists and other researchers for some time [156],

$$T_N^{\omega\nu} = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu} + \frac{1}{4}\eta^{\omega\nu}(R_{\mu\lambda\alpha\beta}R^{\mu\lambda\alpha\beta} - 4R_{\mu\gamma}R^{\mu\gamma} + R^2). \tag{3.11}$$

This energy-momentum tensor is derived from Noether's first theorem, equation (4.20), for the linearized Gauss-Bonnet gravity model [14]. It is the unique, symmetric, gauge invariant, conserved and trace-free expression for the model, all properties of a physical energy momentum tensor as defined by BGRS [29]. This allows for an accurate comparison to be made between the Noether and Hilbert energy-momentum tensors, as in the case of electrodynamics. Since a uniquely defined Noether energy-momentum tensor can be derived from an action with second order derivatives and a second rank tensor potential, of the form $\partial\partial h\partial\partial h$, this model is the ideal candidate to test equivalency with energy-momentum tensor derived from the Hilbert method. We will perform this derivation with free coefficients,

$$\mathcal{L} = AR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + BR_{\nu\beta}R^{\nu\beta} + CR^2, \tag{3.12}$$

in case the reader is interested in the Hilbert energy-momentum tensor for other linearized modified gravity models. Expanding the Lagrangian in Equation (3.12) in terms of Equations (3.8), (3.9) and (3.10) we have,

$$
\begin{aligned}
\mathcal{L} = A(&\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\nu h^{\alpha\beta} - 2\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\alpha h^{\nu\beta} + \partial_\mu\partial_\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h^{\mu\nu}) \\
&+ \frac{1}{4}B(\partial_\mu\partial^\mu h_{\alpha\beta}\partial_\nu\partial^\nu h^{\alpha\beta} + 2\partial_\mu\partial_\nu h^\alpha_\alpha\partial_\beta\partial^\beta h^{\mu\nu} - 4\partial_\mu\partial_\nu h^\nu_\beta\partial_\alpha\partial^\alpha h^{\mu\beta} \\
&+ \partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial^\nu h^\beta_\beta - 4\partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial_\beta h^{\nu\beta} + 2\partial_\mu\partial_\nu h^{\nu\beta}\partial^\mu\partial_\alpha h^\alpha_\beta + 2\partial_\mu\partial_\nu h^\nu_\beta\partial^\beta\partial_\alpha h^{\mu\alpha}) \\
&+ C(\partial_\mu\partial^\mu h^\nu_\nu\partial_\alpha\partial^\alpha h^\beta_\beta - 2\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial^\alpha h^\beta_\beta + \partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}). \quad (3.13)
\end{aligned}
$$

This is the expanded form of the Lagrangian density which we are using to compare the Noether and Hilbert methods. In other words, we are considering a relativistic model in Minkowski spacetime with terms $\partial\partial h\partial\partial h$ in the Lagrangian density. The Hilbert (metric) energy-momentum tensor in Minkowski spacetime has been considered for many models before, for example the spin-2 Fierz-Pauli Lagrangian density [74] (see also [28, 163]), $\mathcal{L}_{FP} = \frac{1}{4}[\partial_\alpha h^\beta_\beta\partial^\alpha h^\gamma_\gamma - \partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} + 2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha} - 2\partial^\alpha h^\beta_\beta\partial^\gamma h_{\gamma\alpha}]$. In [74] Fierz and Pauli developed this action without reference to general metric spacetimes, it is a purely relativistic field theory in Minkowski spacetime. The spin-2 Hilbert energy-momentum tensor was calculated by [28] in their Equation 30. It should be emphasized that $h_{\mu\nu}$ is a symmetric second rank tensor field of a special relativistic (Poincaré invariant) field theory in Minkowski spacetime; these $h_{\mu\nu}$ have no explicit or implicit dependence on the metric $g_{\mu\nu}$. The same goes for the linearized Gauss-Bonnet gravity model. For the purpose of our disproof, this is just a relativistic model in Minkowski spacetime with derivatives of a second rank symmetric tensor potential $h_{\mu\nu}$ in the action (the $\partial\partial h\partial\partial h$ in Equation (3.13)). We use this model because it is sufficiently nontrivial to show that applying both the Noether and Hilbert methods to a common Lagrangian density can yield different results. These results hold more generally for other such nontrival models (higher order derivatives, higher rank of tensor potential), as outlined by the Reasons 1-3 in Section 5. We will now calculate the Hilbert (metric) energy-momentum tensor for this model.

Expressing the Lagrangian in terms of the metric and covariant derivatives

In order to derive the Hilbert energy-momentum tensor, we must replace in Equation (3.13) all ordinary derivatives with covariant derivatives $\partial \to \nabla$, replacing all Minkowski metrics with general metrics $\eta \to g$, and inserting the Jacobian term $\sqrt{-g}$ to this action. In order for brevity, we will write Equation (3.13) compactly as Equation (3.12), thus the Lagrangian takes the form,

$$
\mathcal{L} = A\sqrt{-g}g^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta}R^\nabla_{abcd}R^\nabla_{\mu\nu\alpha\beta} + B\sqrt{-g}g^{\nu b}g^{\beta d}R^\nabla_{\nu\beta}R^\nabla_{bd} + C\sqrt{-g}R^\nabla R^\nabla, \qquad (3.14)
$$

where a superscript ∇ indicates that in the expressions from Equations (3.8), (3.9) and (3.10), the linearized $R^{\mu\nu\alpha\beta}$, $R^{\nu\beta}$ and R have their ordinary derivatives are replaced by covariant derivatives,

$$R^{\nabla}_{\mu\nu\alpha\beta} = \frac{1}{2}(\nabla_\mu\nabla_\alpha h_{\nu\beta} + \nabla_\nu\nabla_\beta h_{\mu\alpha} - \nabla_\mu\nabla_\beta h_{\nu\alpha} - \nabla_\nu\nabla_\alpha h_{\mu\beta}). \tag{3.15}$$

Note that both Latin and Greek indices represent 4 dimensions ($a, b, \cdots = 1, 2, 3, 4$ and $\alpha, \beta, \cdots = 1, 2, 3, 4$). For the Ricci tensor, since it is defined in terms of the Riemann tensor $R_{\nu\beta} = \eta^{\mu\alpha} R_{\mu\nu\alpha\beta}$, we can express the covariant form in terms of the covariant Riemann tensor $R^{\nabla}_{\nu\beta} = g^{\mu\alpha} R^{\nabla}_{\mu\nu\alpha\beta}$. Similarly, the Ricci scalar can be expressed as $R^{\nabla} = g^{\nu\beta} g^{\mu\alpha} R^{\nabla}_{\mu\nu\alpha\beta}$. This allows the Lagrangian to be expressed entirely in terms of the metric and $R^{\nabla}_{\mu\nu\alpha\beta}$ of Equation (3.15),

$$\mathcal{L} = \sqrt{-g}(A g^{a\mu} g^{b\nu} g^{c\alpha} g^{d\beta} + B g^{ac} g^{\mu\alpha} g^{\nu b} g^{\beta d} + C g^{\nu\beta} g^{\mu\alpha} g^{bd} g^{ac}) R^{\nabla}_{\mu\nu\alpha\beta} R^{\nabla}_{abcd}. \tag{3.16}$$

Since we require the Euler derivative for the Hilbert energy-momentum tensor $T^{\gamma\rho}_{H} = \frac{2}{\sqrt{-g}} \frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}\Big|_{g=\eta}$, it is necessary to write $R^{\nabla}_{\mu\nu\alpha\beta}$ in terms of the metric and its derivatives. Therefore we require the second covariant derivatives of the tensor potential $h_{\nu\beta}$,

$$\begin{aligned}\nabla_\mu(\nabla_\alpha h_{\nu\beta}) = {} & \partial_\mu(\partial_\alpha h_{\nu\beta} - \Gamma^{\lambda}_{\alpha\nu} h_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta} h_{\nu\lambda}) - \Gamma^{\lambda}_{\mu\alpha}(\partial_\lambda h_{\nu\beta} - \Gamma^{\rho}_{\lambda\nu} h_{\rho\beta} - \Gamma^{\rho}_{\lambda\beta} h_{\nu\rho}) \\ & - \Gamma^{\lambda}_{\mu\nu}(\partial_\alpha h_{\lambda\beta} - \Gamma^{\rho}_{\alpha\lambda} h_{\rho\beta} - \Gamma^{\rho}_{\alpha\beta} h_{\lambda\rho}) - \Gamma^{\lambda}_{\mu\beta}(\partial_\alpha h_{\nu\lambda} - \Gamma^{\rho}_{\alpha\nu} h_{\rho\lambda} - \Gamma^{\rho}_{\alpha\lambda} h_{\nu\rho}),\end{aligned} \tag{3.17}$$

where $\Gamma^{\lambda}_{\nu\beta} = \frac{1}{2} g^{\mu\lambda}(-\partial_\mu g_{\nu\beta} + \partial_\beta g_{\mu\nu} + \partial_\nu g_{\mu\beta})$ is the Christoffel symbol of the second kind. Since this term appears four times in $R^{\nabla}_{\mu\nu\alpha\beta}$, we are left with,

$$\begin{aligned}R^{\nabla}_{\mu\nu\alpha\beta} = {} & \frac{1}{2}[\partial_\mu(\partial_\alpha h_{\nu\beta} - \Gamma^{\lambda}_{\alpha\nu} h_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta} h_{\nu\lambda}) - \Gamma^{\lambda}_{\mu\alpha}(\partial_\lambda h_{\nu\beta} - \Gamma^{\rho}_{\lambda\nu} h_{\rho\beta} - \Gamma^{\rho}_{\lambda\beta} h_{\nu\rho}) \\ & - \Gamma^{\lambda}_{\mu\nu}(\partial_\alpha h_{\lambda\beta} - \Gamma^{\rho}_{\alpha\lambda} h_{\rho\beta} - \Gamma^{\rho}_{\alpha\beta} h_{\lambda\rho}) - \Gamma^{\lambda}_{\mu\beta}(\partial_\alpha h_{\nu\lambda} - \Gamma^{\rho}_{\alpha\nu} h_{\rho\lambda} - \Gamma^{\rho}_{\alpha\lambda} h_{\nu\rho}) + \partial_\nu(\partial_\beta h_{\mu\alpha} - \Gamma^{\lambda}_{\beta\mu} h_{\lambda\alpha} - \Gamma^{\lambda}_{\beta\alpha} h_{\mu\lambda}) \\ & - \Gamma^{\lambda}_{\nu\beta}(\partial_\lambda h_{\mu\alpha} - \Gamma^{\rho}_{\lambda\mu} h_{\rho\alpha} - \Gamma^{\rho}_{\lambda\alpha} h_{\mu\rho}) - \Gamma^{\lambda}_{\nu\mu}(\partial_\beta h_{\lambda\alpha} - \Gamma^{\rho}_{\beta\lambda} h_{\rho\alpha} - \Gamma^{\rho}_{\beta\alpha} h_{\lambda\rho}) - \Gamma^{\lambda}_{\nu\alpha}(\partial_\beta h_{\mu\lambda} - \Gamma^{\rho}_{\beta\mu} h_{\rho\lambda} - \Gamma^{\rho}_{\beta\lambda} h_{\mu\rho}) \\ & - \partial_\mu(\partial_\beta h_{\nu\alpha} - \Gamma^{\lambda}_{\beta\nu} h_{\lambda\alpha} - \Gamma^{\lambda}_{\beta\alpha} h_{\nu\lambda}) + \Gamma^{\lambda}_{\mu\beta}(\partial_\lambda h_{\nu\alpha} - \Gamma^{\rho}_{\lambda\nu} h_{\rho\alpha} - \Gamma^{\rho}_{\lambda\alpha} h_{\nu\rho}) + \Gamma^{\lambda}_{\mu\nu}(\partial_\beta h_{\lambda\alpha} - \Gamma^{\rho}_{\beta\lambda} h_{\rho\alpha} - \Gamma^{\rho}_{\beta\alpha} h_{\lambda\rho}) \\ & + \Gamma^{\lambda}_{\mu\alpha}(\partial_\beta h_{\nu\lambda} - \Gamma^{\rho}_{\beta\nu} h_{\rho\lambda} - \Gamma^{\rho}_{\beta\lambda} h_{\nu\rho}) - \partial_\nu(\partial_\alpha h_{\mu\beta} - \Gamma^{\lambda}_{\alpha\mu} h_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta} h_{\mu\lambda}) + \Gamma^{\lambda}_{\nu\alpha}(\partial_\lambda h_{\mu\beta} - \Gamma^{\rho}_{\lambda\mu} h_{\rho\beta} - \Gamma^{\rho}_{\lambda\beta} h_{\mu\rho}) \\ & + \Gamma^{\lambda}_{\nu\mu}(\partial_\alpha h_{\lambda\beta} - \Gamma^{\rho}_{\alpha\lambda} h_{\rho\beta} - \Gamma^{\rho}_{\alpha\beta} h_{\lambda\rho}) + \Gamma^{\lambda}_{\nu\beta}(\partial_\alpha h_{\mu\lambda} - \Gamma^{\rho}_{\alpha\mu} h_{\rho\lambda} - \Gamma^{\rho}_{\alpha\lambda} h_{\mu\rho})].\end{aligned} \tag{3.18}$$

Expanding this expression is a bit tedious. Many terms cancel, and combine. Familiar terms here, the Riemann tensor $\bar{R}^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$ and linearized Christoffel symbol $\bar{\Gamma}_{\lambda\mu\alpha} = \frac{1}{2}(-\partial_\lambda h_{\mu\alpha} + \partial_\alpha h_{\lambda\mu} + \partial_\mu h_{\lambda\alpha})$, allow for Equation (3.18) to be expressed much more compactly as,

$$R^{\nabla}_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{2}\bar{R}^{\lambda}_{\alpha\mu\nu}h_{\lambda\beta} + \frac{1}{2}\bar{R}^{\lambda}_{\beta\mu\nu}h_{\lambda\alpha} - 2\Gamma^{\lambda}_{\nu\alpha}\bar{\Gamma}_{\lambda\mu\beta} - 2\Gamma^{\lambda}_{\mu\beta}\bar{\Gamma}_{\lambda\nu\alpha} + 2\Gamma^{\lambda}_{\mu\alpha}\bar{\Gamma}_{\lambda\nu\beta} + 2\Gamma^{\lambda}_{\nu\beta}\bar{\Gamma}_{\lambda\mu\alpha} + \Gamma^{\lambda}_{\mu\beta}\Gamma^{\rho}_{\alpha\nu}h_{\rho\lambda} - \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\rho}_{\beta\nu}h_{\rho\lambda}. \quad (3.19)$$

We now make an important note that can save the reader many pages of calculations. The Hilbert energy-momentum tensor requires us to replace the metric tensor with the Minkowski metric after variation $g \to \eta$. Therefore any derivatives of the metric that remain after variation will be zero upon differentiation. Some terms in the Lagrangian, namely those of the form $\Gamma\Gamma$ (not $\bar{\Gamma}$, because these are the linearized expressions) will all vanish upon $g \to \eta$. Due to this fact we will neglect such terms from $R^{\nabla}_{\mu\nu\alpha\beta}$, as they will not contribute to the final result. For clarity this will be labelled $R^{H\nabla}_{\mu\nu\alpha\beta}$ for the terms which contribute to the Hilbert energy-momentum tensor,

$$R^{H\nabla}_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{2}(\partial_{\mu}\Gamma^{\lambda}_{\nu\alpha} - \partial_{\nu}\Gamma^{\lambda}_{\mu\alpha})h_{\lambda\beta} + \frac{1}{2}(\partial_{\mu}\Gamma^{\lambda}_{\nu\beta} - \partial_{\nu}\Gamma^{\lambda}_{\mu\beta})h_{\lambda\alpha} - 2\Gamma^{\lambda}_{\nu\alpha}\bar{\Gamma}_{\lambda\mu\beta} - 2\Gamma^{\lambda}_{\mu\beta}\bar{\Gamma}_{\lambda\nu\alpha} + 2\Gamma^{\lambda}_{\mu\alpha}\bar{\Gamma}_{\lambda\nu\beta} + 2\Gamma^{\lambda}_{\nu\beta}\bar{\Gamma}_{\lambda\mu\alpha}. \quad (3.20)$$

This expression can be further 'simplified' by noting that the $R^{H\nabla}_{\mu\nu\alpha\beta}$ is multiplied by $R^{H\nabla}_{abcd}$. The vast majority of terms in this expansion will have a $\Gamma\Gamma$ contribution. Therefore keeping only those which will be nonzero after variation for $R^{H\nabla}_{\mu\nu\alpha\beta}R^{H\nabla}_{abcd}$, we are left with the Lagrangian,

$$\begin{aligned}\mathcal{L} = \sqrt{-g}(&Ag^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + Bg^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + Cg^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(R_{\mu\nu\alpha\beta}R_{abcd}\\ &- 2\Gamma^{\lambda}_{\nu\alpha}R_{abcd}\bar{\Gamma}_{\lambda\mu\beta} - 2\Gamma^{\lambda}_{\mu\beta}R_{abcd}\bar{\Gamma}_{\lambda\nu\alpha} + 2\Gamma^{\lambda}_{\mu\alpha}R_{abcd}\bar{\Gamma}_{\lambda\nu\beta} + 2\Gamma^{\lambda}_{\nu\beta}R_{abcd}\bar{\Gamma}_{\lambda\mu\alpha}\\ &- 2\Gamma^{\gamma}_{bc}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\gamma ad} - 2\Gamma^{\gamma}_{ad}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\gamma bc} + 2\Gamma^{\gamma}_{ac}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\gamma bd} + 2\Gamma^{\gamma}_{bd}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\gamma ac}\\ &- \frac{1}{2}\partial_{\mu}\Gamma^{\lambda}_{\nu\alpha}R_{abcd}h_{\lambda\beta} + \frac{1}{2}\partial_{\nu}\Gamma^{\lambda}_{\mu\alpha}R_{abcd}h_{\lambda\beta} + \frac{1}{2}\partial_{\mu}\Gamma^{\lambda}_{\nu\beta}R_{abcd}h_{\lambda\alpha} - \frac{1}{2}\partial_{\nu}\Gamma^{\lambda}_{\mu\beta}R_{abcd}h_{\lambda\alpha}\\ &- \frac{1}{2}\partial_{a}\Gamma^{\gamma}_{bc}R_{\mu\nu\alpha\beta}h_{\gamma d} + \frac{1}{2}\partial_{b}\Gamma^{\gamma}_{ac}R_{\mu\nu\alpha\beta}h_{\gamma d} + \frac{1}{2}\partial_{a}\Gamma^{\gamma}_{bd}R_{\mu\nu\alpha\beta}h_{\gamma c} - \frac{1}{2}\partial_{b}\Gamma^{\gamma}_{ad}R_{\mu\nu\alpha\beta}h_{\gamma c}). \quad (3.21)\end{aligned}$$

The Lagrangian terms are sorted as follows. The first line is terms which will be nonzero after differentiation $\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}}$ and $g \to \eta$, the second and third lines will be nonzero after $\partial_{\omega}\frac{\partial\mathcal{L}}{\partial(\partial_{\omega}g_{\gamma\rho})}$ and $g \to \eta$, the fourth and fifth lines will be nonzero after $\partial_{\xi}\partial_{\omega}\frac{\partial\mathcal{L}}{\partial(\partial_{\xi}\partial_{\omega}g_{\gamma\rho})}$ and $g \to \eta$.

Taking the Euler derivative of the Lagrangian

Recall that we require the Euler derivative $\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}} = \frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} - \partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} + \partial_\xi\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})}$ in order to derive the Hilbert energy-momentum tensor. Performing the differentiation of the relevant parts, first with respect to the metric, the nonzero terms after $g \to \eta$ are,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} &= \frac{\partial\sqrt{-g}}{\partial g_{\gamma\rho}}[Ag^{\mu a}g^{\nu b}g^{\alpha c}g^{\beta d} + Bg^{\nu b}g^{\beta d}g^{\mu\alpha}g^{ac} + Cg^{\mu\alpha}g^{\nu\beta}g^{ac}g^{bd}](R_{\mu\nu\alpha\beta}R_{abcd})\\
&+ \sqrt{-g}[A\frac{\partial g^{\mu a}}{\partial g_{\gamma\rho}}g^{\nu b}g^{\alpha c}g^{\beta d} + Ag^{\mu a}\frac{\partial g^{\nu b}}{\partial g_{\gamma\rho}}g^{\alpha c}g^{\beta d} + Ag^{\mu a}g^{\nu b}\frac{\partial g^{\alpha c}}{\partial g_{\gamma\rho}}g^{\beta d} + Ag^{\mu a}g^{\nu b}g^{\alpha c}\frac{\partial g^{\beta d}}{\partial g_{\gamma\rho}}\\
&+ B\frac{\partial g^{\nu b}}{\partial g_{\gamma\rho}}g^{\beta d}g^{\mu\alpha}g^{ac} + Bg^{\nu b}\frac{\partial g^{\beta d}}{\partial g_{\gamma\rho}}g^{\mu\alpha}g^{ac} + Bg^{\nu b}g^{\beta d}\frac{\partial g^{\mu\alpha}}{\partial g_{\gamma\rho}}g^{ac} + Bg^{\nu b}g^{\beta d}g^{\mu\alpha}\frac{\partial g^{ac}}{\partial g_{\gamma\rho}}\\
&+ C\frac{\partial g^{\mu\alpha}}{\partial g_{\gamma\rho}}g^{\nu\beta}g^{ac}g^{bd} + Cg^{\mu\alpha}\frac{\partial g^{\nu\beta}}{\partial g_{\gamma\rho}}g^{ac}g^{bd} + Cg^{\mu\alpha}g^{\nu\beta}\frac{\partial g^{ac}}{\partial g_{\gamma\rho}}g^{bd} + Cg^{\mu\alpha}g^{\nu\beta}g^{ac}\frac{\partial g^{bd}}{\partial g_{\gamma\rho}}](R_{\mu\nu\alpha\beta}R_{abcd}).\end{aligned} \tag{3.22}$$

Inserting $\frac{\partial\sqrt{-g}}{\partial g_{\gamma\rho}}$ and $\frac{\partial g^{\lambda\nu}}{\partial g_{\beta\gamma}}$, and expanding all brackets, we are left with the following expression,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} &= -\frac{1}{2}\sqrt{-g}(-Ag^{\gamma\rho}g^{\mu a}g^{\nu b}g^{\alpha c}g^{\beta d} + Ag^{\gamma\mu}g^{\rho a}g^{\nu b}g^{\alpha c}g^{\beta d} + Ag^{\rho\mu}g^{\gamma a}g^{\nu b}g^{\alpha c}g^{\beta d}\\
&+ Ag^{\gamma b}g^{\rho\nu}g^{\mu a}g^{\alpha c}g^{\beta d} + Ag^{\rho b}g^{\gamma\nu}g^{\mu a}g^{\alpha c}g^{\beta d} + Ag^{\gamma c}g^{\rho\alpha}g^{\mu a}g^{\nu b}g^{\beta d} + Ag^{\rho c}g^{\gamma\alpha}g^{\mu a}g^{\nu b}g^{\beta d}\\
&+ Ag^{\gamma d}g^{\rho\beta}g^{\mu a}g^{\nu b}g^{\alpha c} + Ag^{\rho d}g^{\gamma\beta}g^{\mu a}g^{\nu b}g^{\alpha c} - Bg^{\gamma\rho}g^{\nu b}g^{\beta d}g^{\mu\alpha}g^{ac} + Bg^{\gamma b}g^{\rho\nu}g^{\beta d}g^{\mu\alpha}g^{ac}\\
&+ Bg^{\rho b}g^{\gamma\nu}g^{\beta d}g^{\mu\alpha}g^{ac} + Bg^{\gamma d}g^{\rho\beta}g^{\nu b}g^{\mu\alpha}g^{ac} + Bg^{\rho d}g^{\gamma\beta}g^{\nu b}g^{\mu\alpha}g^{ac} + Bg^{\gamma\mu}g^{\rho\alpha}g^{\nu b}g^{\beta d}g^{ac}\\
&+ Bg^{\rho\mu}g^{\gamma\alpha}g^{\nu b}g^{\beta d}g^{ac} + Bg^{\gamma c}g^{\rho a}g^{\nu b}g^{\beta d}g^{\mu\alpha} + Bg^{\rho c}g^{\gamma a}g^{\nu b}g^{\beta d}g^{\mu\alpha} - Cg^{\gamma\rho}g^{\mu\alpha}g^{\nu\beta}g^{ac}g^{bd}\\
&+ Cg^{\gamma\mu}g^{\rho\alpha}g^{\nu\beta}g^{ac}g^{bd} + Cg^{\rho\mu}g^{\gamma\alpha}g^{\nu\beta}g^{ac}g^{bd} + Cg^{\gamma\beta}g^{\rho\nu}g^{\mu\alpha}g^{ac}g^{bd} + Cg^{\rho\beta}g^{\gamma\nu}g^{\mu\alpha}g^{ac}g^{bd}\\
&+ Cg^{\gamma c}g^{\rho a}g^{\mu\alpha}g^{\nu\beta}g^{bd} + Cg^{\rho c}g^{\gamma a}g^{\mu\alpha}g^{\nu\beta}g^{bd} + Cg^{\gamma d}g^{\rho b}g^{\mu\alpha}g^{\nu\beta}g^{ac} + Cg^{\rho d}g^{\gamma b}g^{\mu\alpha}g^{\nu\beta}g^{ac})(R_{\mu\nu\alpha\beta}R_{abcd}).\end{aligned} \tag{3.23}$$

To write this expression more compactly we express the terms proportional to A as $\bar{g}_A^{\gamma\rho\mu a\nu b\alpha c\beta d} = -g^{\gamma\rho}g^{\mu a}g^{\nu b}g^{\alpha c}g^{\beta d} + g^{\gamma\mu}g^{\rho a}g^{\nu b}g^{\alpha c}g^{\beta d} + g^{\rho\mu}g^{\gamma a}g^{\nu b}g^{\alpha c}g^{\beta d} + g^{\gamma b}g^{\rho\nu}g^{\mu a}g^{\alpha c}g^{\beta d} + g^{\rho b}g^{\gamma\nu}g^{\mu a}g^{\alpha c}g^{\beta d} + g^{\gamma c}g^{\rho\alpha}g^{\mu a}g^{\nu b}g^{\beta d} + g^{\rho c}g^{\gamma\alpha}g^{\mu a}g^{\nu b}g^{\beta d} + g^{\gamma d}g^{\rho\beta}g^{\mu a}g^{\nu b}g^{\alpha c} + g^{\rho d}g^{\gamma\beta}g^{\mu a}g^{\nu b}g^{\alpha c}$,

we express the terms proportional to B as $\bar{g}_B^{\gamma\rho\nu b\beta d\mu\alpha ac} = -g^{\gamma\rho}g^{\nu b}g^{\beta d}g^{\mu\alpha}g^{ac} + g^{\gamma b}g^{\rho\nu}g^{\beta d}g^{\mu\alpha}g^{ac} + g^{\rho b}g^{\gamma\nu}g^{\beta d}g^{\mu\alpha}g^{ac} + g^{\gamma d}g^{\rho\beta}g^{\nu b}g^{\mu\alpha}g^{ac} + g^{\rho d}g^{\gamma\beta}g^{\nu b}g^{\mu\alpha}g^{ac} + g^{\gamma\mu}g^{\rho\alpha}g^{\nu b}g^{\beta d}g^{ac} + g^{\rho\mu}g^{\gamma\alpha}g^{\nu b}g^{\beta d}g^{ac} + g^{\gamma c}g^{\rho a}g^{\nu b}g^{\beta d}g^{\mu\alpha} + g^{\rho c}g^{\gamma a}g^{\nu b}g^{\beta d}g^{\mu\alpha}$

and we express the terms proportional to C as $\bar{g}_C^{\gamma\rho\mu\alpha\nu\beta acbd} = -g^{\gamma\rho}g^{\mu\alpha}g^{\nu\beta}g^{ac}g^{bd} + g^{\gamma\mu}g^{\rho\alpha}g^{\nu\beta}g^{ac}g^{bd} + g^{\rho\mu}g^{\gamma\alpha}g^{\nu\beta}g^{ac}g^{bd} + g^{\gamma\beta}g^{\rho\nu}g^{\mu\alpha}g^{ac}g^{bd} + g^{\rho\beta}g^{\gamma\nu}g^{\mu\alpha}g^{ac}g^{bd} + g^{\gamma c}g^{\rho a}g^{\mu\alpha}g^{\nu\beta}g^{bd} + g^{\rho c}g^{\gamma a}g^{\mu\alpha}g^{\nu\beta}g^{bd} + g^{\gamma d}g^{\rho b}g^{\mu\alpha}g^{\nu\beta}g^{ac} + g^{\rho d}g^{\gamma b}g^{\mu\alpha}g^{\nu\beta}g^{ac}$.

Therefore the derivative of the Lagrangian with respect to the metric is expressed compactly as,

$$\frac{\partial \mathcal{L}}{\partial g_{\gamma\rho}} = -\frac{1}{2}\sqrt{-g}(A\bar{g}_A^{\gamma\rho\mu a\nu b\alpha c\beta d} + B\bar{g}_B^{\gamma\rho\nu b\beta d\mu\alpha ac} + C\bar{g}_C^{\gamma\rho\mu\alpha\nu\beta acbd})(R_{\mu\nu\alpha\beta}R_{abcd}). \tag{3.24}$$

Next we will differentiate the Lagrangian in Equation (3.21) with respect to derivatives of the metric. Only the second and third lines of the Lagrangian in Equation (3.21) will be nonzero after $\frac{\partial \mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})}$, as only terms with a non-linearized Christoffel symbol $\Gamma^\lambda_{\nu\alpha} = \frac{1}{2}g^{m\lambda}(-\partial_m g_{\nu\alpha} + \partial_\alpha g_{m\nu} + \partial_\nu g_{m\alpha})$ have linear in ∂g contributions. These terms will be differentiating as $\frac{\partial\partial_m g_{\nu\alpha}}{\partial(\partial_\omega g_{\gamma\rho})} = \delta^\omega_m \Delta^{\gamma\rho}_{\nu\alpha}$ where $\Delta^{\gamma\rho}_{\nu\alpha} = \frac{1}{2}(\delta^\gamma_\nu\delta^\rho_\alpha + \delta^\gamma_\alpha\delta^\rho_\nu)$. Differentiating the Chrisoffel symbol therefore yields,

$$\frac{\partial \Gamma^\lambda_{\nu\alpha}}{\partial(\partial_\omega g_{\gamma\rho})} = \frac{1}{2}g^{m\lambda}(-\delta^\omega_m\Delta^{\gamma\rho}_{\nu\alpha} + \delta^\omega_\alpha\Delta^{\gamma\rho}_{m\nu} + \delta^\omega_\nu\Delta^{\gamma\rho}_{m\alpha}) = \frac{1}{2}g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\nu\alpha}, \tag{3.25}$$

where above to abbreviate we call the combination of the Kronecker deltas in brackets $\bar{\Delta}^{\omega\gamma\rho}_{m\nu\alpha} = -\delta^\omega_m\Delta^{\gamma\rho}_{\nu\alpha} + \delta^\omega_\alpha\Delta^{\gamma\rho}_{m\nu} + \delta^\omega_\nu\Delta^{\gamma\rho}_{m\alpha}$. Using this compact notiation the derivative of the Lagrangian with respect to derivatives of the metric is,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} = {}& \sqrt{-g}(Ag^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + Bg^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + Cg^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(\\ & - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\nu\alpha}R_{abcd}\bar{\Gamma}_{\lambda\mu\beta} - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\mu\beta}R_{abcd}\bar{\Gamma}_{\lambda\nu\alpha} + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\mu\alpha}R_{abcd}\bar{\Gamma}_{\lambda\nu\beta} + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\nu\beta}R_{abcd}\bar{\Gamma}_{\lambda\mu\alpha} \\ & - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbc}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ad} - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mad}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bc} + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mac}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bd} + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbd}R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ac}).\end{aligned} \tag{3.26}$$

Since we require $\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})}$ we must differentiate the above expression with by ∂_ω. This process is in general quite messy, but since any ∂g will be zero upon $g \to \eta$, only the linearized Riemann tensor and linearized Christoffel symbol will, differentiated, give rise to nonzero contributions,

$$\begin{aligned}\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} = {}& 2\sqrt{-g}(Ag^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + Bg^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + Cg^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(\\ & -g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\nu\alpha}\partial_\omega[R_{abcd}\bar{\Gamma}_{\lambda\mu\beta}] - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\mu\beta}\partial_\omega[R_{abcd}\bar{\Gamma}_{\lambda\nu\alpha}] + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\mu\alpha}\partial_\omega[R_{abcd}\bar{\Gamma}_{\lambda\nu\beta}] + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{m\nu\beta}\partial_\omega[R_{abcd}\bar{\Gamma}_{\lambda\mu\alpha}]),\end{aligned} \tag{3.27}$$

where the final two lines in Equation (3.26) were combined by interchange $abcd \leftrightarrow \mu\nu\alpha\beta$.

Finally for the terms proportional to $\partial\partial g$ in the fourth and fifth lines of the Lagrangian in Equation (3.21), we require the differentiated Christoffel symbol $\partial_a \Gamma^{\lambda}_{bc} = \frac{1}{2}\partial_a g^{m\lambda}(-\partial_m g_{bc} + \partial_c g_{mb} + \partial_b g_{mc}) + \frac{1}{2} g^{m\lambda}\partial_a(-\partial_m g_{bc} + \partial_c g_{mb} + \partial_b g_{mc})$. The first term will be zero upon $g \to \eta$ so we can neglect it, leaving $\partial_a \Gamma^{\lambda}_{bc} = \frac{1}{2} g^{m\lambda}(-\partial_a\partial_m g_{bc} + \partial_a\partial_c g_{mb} + \partial_a\partial_b g_{mc})$. Differentiating each term will yields $\frac{\partial\partial_a\partial_m g_{bc}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} = \Delta^{\xi\omega}_{am}\Delta^{\gamma\rho}_{bc}$. Therefore differentiating of the derivative of the Christoffel symbol gives,

$$\frac{\partial\partial_a \Gamma^{\lambda}_{bc}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} = \frac{1}{2} g^{m\lambda}(-\Delta^{\xi\omega}_{am}\Delta^{\gamma\rho}_{bc} + \Delta^{\xi\omega}_{ac}\Delta^{\gamma\rho}_{mb} + \Delta^{\xi\omega}_{ab}\Delta^{\gamma\rho}_{mc}) = \frac{1}{2} g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}. \tag{3.28}$$

The above expression in brackets was abbreviated with $\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc} = -\Delta^{\xi\omega}_{am}\Delta^{\gamma\rho}_{bc} + \Delta^{\xi\omega}_{ac}\Delta^{\gamma\rho}_{mb} + \Delta^{\xi\omega}_{ab}\Delta^{\gamma\rho}_{mc}$. Using this compact notation the derivative of the Lagrangian with respect to two derivatives of the metric is,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} = \frac{1}{4}&\sqrt{-g}(A g^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + B g^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + C g^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(\\ &- g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{\mu m\nu\alpha}R_{abcd}h_{\lambda\beta} + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{\nu m\mu\alpha}R_{abcd}h_{\lambda\beta} + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{\mu m\nu\beta}R_{abcd}h_{\lambda\alpha} - g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{\nu m\mu\beta}R_{abcd}h_{\lambda\alpha} \\ &- g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}R_{\mu\nu\alpha\beta}h_{\lambda d} + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}R_{\mu\nu\alpha\beta}h_{\lambda d} + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}R_{\mu\nu\alpha\beta}h_{\lambda c} - g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmad}R_{\mu\nu\alpha\beta}h_{\lambda c}).\end{aligned} \tag{3.29}$$

Since we require $\partial_\xi\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})}$ we must differentiate the above expression by $\partial_\xi\partial_\omega$. Again this process is in general quite messy, but since any ∂g will be zero upon $g \to \eta$, only the linearized Riemann tensor and the potential $h_{\lambda\alpha}$, differentiated, give rise to nonzero contributions,

$$\begin{aligned}\partial_\xi\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})} = \frac{1}{2}&\sqrt{-g}(A g^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + B g^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + C g^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(\\ &- g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] \\ &+ g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}] - g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmad}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}]),\end{aligned} \tag{3.30}$$

where the bottom two lines in $\frac{\partial\mathcal{L}}{\partial(\partial_\xi\partial_\omega g_{\gamma\rho})}$ were combined by interchange $abcd \leftrightarrow \mu\nu\alpha\beta$. Therefore for the total Euler derivative we have, combining equations (3.24), (3.27) and (3.30),

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta g_{\gamma\rho}} = -\frac{1}{2}\sqrt{-g}\Bigg(& (A\bar{g}_A^{\gamma\rho\mu\alpha\nu b\alpha c\beta d} + B\bar{g}_B^{\gamma\rho\nu b\beta d\mu\alpha ac} + C\bar{g}_C^{\gamma\rho\mu\alpha\nu\beta acbd})(R_{\mu\nu\alpha\beta}R_{abcd}) \\
& + 4(Ag^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + Bg^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + Cg^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(-g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbc}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ad}] \\
& - g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mad}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bc}] + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mac}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bd}] + g^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbd}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ac}]) \\
& - (Ag^{a\mu}g^{b\nu}g^{c\alpha}g^{d\beta} + Bg^{ac}g^{\mu\alpha}g^{\nu b}g^{\beta d} + Cg^{\nu\beta}g^{\mu\alpha}g^{bd}g^{ac})(-g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] \\
& + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] + g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}] - g^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmad}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}])\Bigg).
\end{aligned} \tag{3.31}
$$

The Hilbert energy-momentum tensor

We can now turn our attention to the Hilbert energy-momentum tensor $T_H^{\gamma\rho}$ in Equation (3.49). Since we have calculated the Euler derivative in the previous section, evaluating this expression for $g \to \eta$ yields,

$$
\begin{aligned}
T_H^{\gamma\rho} = & -(A\bar{\eta}_A^{\gamma\rho\mu\alpha\nu b\alpha c\beta d} + B\bar{\eta}_B^{\gamma\rho\nu b\beta d\mu\alpha ac} + C\bar{\eta}_C^{\gamma\rho\mu\alpha\nu\beta acbd})(R_{\mu\nu\alpha\beta}R_{abcd}) \\
& - 4(A\eta^{a\mu}\eta^{b\nu}\eta^{c\alpha}\eta^{d\beta} + B\eta^{ac}\eta^{\mu\alpha}\eta^{\nu b}\eta^{\beta d} + C\eta^{\nu\beta}\eta^{\mu\alpha}\eta^{bd}\eta^{ac})(-\eta^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbc}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ad}] \\
& - \eta^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mad}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bc}] + \eta^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mac}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda bd}] + \eta^{m\lambda}\bar{\Delta}^{\omega\gamma\rho}_{mbd}\partial_\omega[R_{\mu\nu\alpha\beta}\bar{\Gamma}_{\lambda ac}]) \\
& + (A\eta^{a\mu}\eta^{b\nu}\eta^{c\alpha}\eta^{d\beta} + B\eta^{ac}\eta^{\mu\alpha}\eta^{\nu b}\eta^{\beta d} + C\eta^{\nu\beta}\eta^{\mu\alpha}\eta^{bd}\eta^{ac})(-\eta^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] \\
& + \eta^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda d}] + \eta^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}] - \eta^{m\lambda}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmad}\partial_\xi\partial_\omega[R_{\mu\nu\alpha\beta}h_{\lambda c}]),
\end{aligned} \tag{3.32}
$$

where the $\bar{g}_A^{\gamma\rho\mu\alpha\nu b\alpha c\beta d} \to \bar{\eta}_A^{\gamma\rho\mu\alpha\nu b\alpha c\beta d}$ is the same form with the metric tensor replaced by the Minkowski metric. If we contract all visible Minkowski tensors and the $\bar{\eta}_A^{\gamma\rho\mu\alpha\nu b\alpha c\beta d}$, then we obtain,

$$
\begin{aligned}
T_H^{\gamma\rho} = & -A(-\eta^{\gamma\rho}R^{abcd}R_{abcd}+8R^{\gamma bcd}R^{\rho}{}_{bcd})-B(-\eta^{\gamma\rho}R^{bd}R_{bd}+4R^{\gamma d}R^{\rho}_{d}+4R^{\gamma b\rho d}R_{bd})-C(-\eta^{\gamma\rho}R^2+8R^{\gamma\rho}R) \\
& - 16A\bar{\Delta}^{\omega\gamma\rho}_{mac}\partial_\omega[R^{abcd}\bar{\Gamma}^{m}{}_{bd}] - 8C(-\bar{\Delta}^{\omega\gamma\rho}_{mad}\partial_\omega[R\bar{\Gamma}^{mda}] + \eta^{ad}\bar{\Delta}^{\omega\gamma\rho}_{mad}\partial_\omega[R\bar{\Gamma}^{mb}{}_{b}]) \\
& - 4B(-\bar{\Delta}^{\omega\gamma\rho}_{mba}\partial_\omega[R^{bd}\bar{\Gamma}^{ma}{}_{d}] - \bar{\Delta}^{\omega\gamma\rho}_{mad}\partial_\omega[R^{bd}\bar{\Gamma}^{m}{}_{b}{}^{a}] + \eta^{ac}\bar{\Delta}^{\omega\gamma\rho}_{mac}\partial_\omega[R^{bd}\bar{\Gamma}^{m}{}_{bd}] + \bar{\Delta}^{\omega\gamma\rho}_{mbd}\partial_\omega[R^{bd}\bar{\Gamma}^{mc}{}_{c}]) \\
& + 4A\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}\partial_\xi\partial_\omega[R^{abcd}h^{m}_{c}] + 2C(-\eta^{ac}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}\partial_\xi\partial_\omega[Rh^{mb}] + \eta^{ac}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}\partial_\xi\partial_\omega[Rh^{mb}]) \\
& + B(-\eta^{ac}\hat{\Delta}^{\xi\omega\gamma\rho}_{ambc}\partial_\xi\partial_\omega[R^{bd}h^{m}_{d}]+\eta^{ac}\hat{\Delta}^{\xi\omega\gamma\rho}_{bmac}\partial_\xi\partial_\omega[R^{bd}h^{m}_{d}]+\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}\partial_\xi\partial_\omega[R^{bd}h^{ma}]-\hat{\Delta}^{\xi\omega\gamma\rho}_{bmad}\partial_\xi\partial_\omega[R^{bd}h^{ma}]).
\end{aligned} \tag{3.33}
$$

Next we will contract all of the $\bar{\Delta}^{\omega\gamma\rho}_{mba}$ and $\hat{\Delta}^{\xi\omega\gamma\rho}_{ambd}$,

$$
\begin{aligned}
T_H^{\gamma\rho} = -A(-\eta^{\gamma\rho}R^{abcd}R_{abcd}+8R^{\gamma bcd}R^{\rho}{}_{bcd})-B(-\eta^{\gamma\rho}R^{bd}R_{bd}+4R^{\gamma d}R^{\rho}_{d}+4R^{\gamma b\rho d}R_{bd})-C(-\eta^{\gamma\rho}R^2+8R^{\gamma\rho}R)
\end{aligned}
$$

$$
\begin{aligned}
&+ 16A\partial_\omega[R^{\gamma b\rho d}\bar{\Gamma}^{\omega}{}_{bd} - R^{\rho b\omega d}\bar{\Gamma}^{\gamma}{}_{bd} - R^{\gamma b\omega d}\bar{\Gamma}^{\rho}{}_{bd}]\\
&+ 4B\partial_\omega[-R^{\rho d}\bar{\Gamma}^{\omega\gamma}{}_{d} + \eta^{\gamma\rho}R^{bd}\bar{\Gamma}^{\omega}{}_{bd} + R^{\gamma\rho}\bar{\Gamma}^{\omega b}{}_{b} - R^{\gamma d}\bar{\Gamma}^{\omega\rho}{}_{d} + R^{\omega d}\bar{\Gamma}^{\gamma\rho}{}_{d} - \eta^{\omega\rho}R^{bd}\bar{\Gamma}^{\gamma}{}_{bd}]\\
&+ 4B\partial_\omega[-R^{\rho\omega}\bar{\Gamma}^{\gamma b}{}_{b} + R^{\omega d}\bar{\Gamma}^{\rho\gamma}{}_{d} - \eta^{\omega\gamma}R^{bd}\bar{\Gamma}^{\rho}{}_{bd} - R^{\gamma\omega}\bar{\Gamma}^{\rho b}{}_{b} + R^{\rho d}\bar{\Gamma}^{\gamma\omega}{}_{d} + R^{\gamma d}\bar{\Gamma}^{\rho\omega}{}_{d}]\\
&+ 8C\partial_\omega[-R\bar{\Gamma}^{\omega\rho\gamma} + \eta^{\gamma\rho}R\bar{\Gamma}^{\omega b}{}_{b} + R\bar{\Gamma}^{\gamma\omega\rho} - \eta^{\omega\rho}R\bar{\Gamma}^{\gamma b}{}_{b} + R\bar{\Gamma}^{\rho\omega\gamma} - \eta^{\omega\gamma}R\bar{\Gamma}^{\rho b}{}_{b}]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2A\partial_a\partial_\omega[-R^{a\gamma\rho d}h^{\omega}_{\ d} - R^{a\rho\gamma d}h^{\omega}_{\ d} + R^{a\rho\omega d}h^{\gamma}_{\ d} + R^{a\gamma\omega d}h^{\rho}_{\ d} + R^{a\omega\rho d}h^{\gamma}_{\ d} + R^{a\omega\gamma d}h^{\rho}_{\ d}]\\
&+ B\partial_a\partial_\omega[\eta^{\gamma\rho}R^{ad}h^{\omega}_{\ d} - \eta^{\rho\omega}R^{ad}h^{\gamma}_{\ d} - \eta^{\gamma\omega}R^{ad}h^{\rho}_{\ d} + R^{a\omega}h^{\gamma\rho} + R^{\gamma\rho}h^{\omega a} - R^{\rho\omega}h^{\gamma a} - R^{\gamma\omega}h^{\rho a}]\\
&+ \frac{1}{2}B\partial_a\partial_\omega[-\eta^{a\rho}R^{\gamma d}h^{\omega}_{\ d} - \eta^{a\gamma}R^{\rho d}h^{\omega}_{\ d} + \eta^{a\omega}R^{\rho d}h^{\gamma}_{\ d} + \eta^{a\omega}R^{\gamma d}h^{\rho}_{\ d} + \eta^{a\rho}R^{\omega d}h^{\gamma}_{\ d} + \eta^{a\gamma}R^{\omega d}h^{\rho}_{\ d}]\\
&+ 2C\partial_a\partial_\omega[\eta^{a\omega}Rh^{\gamma\rho} + \eta^{\gamma\rho}Rh^{\omega a} - \eta^{\rho\omega}Rh^{\gamma a} - \eta^{\gamma\omega}Rh^{\rho a}]. \quad (3.34)
\end{aligned}
$$

Separating the part proportional to $\eta^{\gamma\rho}$,

$$
\begin{aligned}
T_H^{\gamma\rho} = \eta^{\gamma\rho}\Big(&AR^{abcd}R_{abcd} + C(R^2 + 8\partial_\omega[R\bar{\Gamma}^{\omega b}{}_{b}] + 2\partial_a\partial_\omega[Rh^{\omega a}])\\
&+ B(R^{bd}R_{bd} + 4\partial_\omega[R^{bd}\bar{\Gamma}^{\omega}{}_{bd}] + \partial_a\partial_\omega[R^{ad}h^{\omega}_{\ d}])\Big)
\end{aligned}
$$

$$
- 8AR^{\gamma bcd}R^{\rho}{}_{bcd} - B(4R^{\gamma d}R^{\rho}_{d} + 4R^{\gamma b\rho d}R_{bd}) - 8CR^{\gamma\rho}R
$$

$$
\begin{aligned}
&+ 16A\partial_\omega[R^{\gamma b\rho d}\bar{\Gamma}^{\omega}{}_{bd} - R^{\rho b\omega d}\bar{\Gamma}^{\gamma}{}_{bd} - R^{\gamma b\omega d}\bar{\Gamma}^{\rho}{}_{bd}]\\
&+ 4B\partial_\omega[-R^{\rho d}\bar{\Gamma}^{\omega\gamma}{}_{d} + R^{\gamma\rho}\bar{\Gamma}^{\omega b}{}_{b} - R^{\gamma d}\bar{\Gamma}^{\omega\rho}{}_{d} + R^{\omega d}\bar{\Gamma}^{\gamma\rho}{}_{d} - \eta^{\omega\rho}R^{bd}\bar{\Gamma}^{\gamma}{}_{bd}]\\
&+ 4B\partial_\omega[-R^{\rho\omega}\bar{\Gamma}^{\gamma b}{}_{b} + R^{\omega d}\bar{\Gamma}^{\rho\gamma}{}_{d} - \eta^{\omega\gamma}R^{bd}\bar{\Gamma}^{\rho}{}_{bd} - R^{\gamma\omega}\bar{\Gamma}^{\rho b}{}_{b} + R^{\rho d}\bar{\Gamma}^{\gamma\omega}{}_{d} + R^{\gamma d}\bar{\Gamma}^{\rho\omega}{}_{d}]\\
&+ 8C\partial_\omega[-R\bar{\Gamma}^{\omega\rho\gamma} + R\bar{\Gamma}^{\gamma\omega\rho} - \eta^{\omega\rho}R\bar{\Gamma}^{\gamma b}{}_{b} + R\bar{\Gamma}^{\rho\omega\gamma} - \eta^{\omega\gamma}R\bar{\Gamma}^{\rho b}{}_{b}]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2A\partial_a\partial_\omega[-R^{a\gamma\rho d}h^{\omega}_{\ d} - R^{a\rho\gamma d}h^{\omega}_{\ d} + R^{a\rho\omega d}h^{\gamma}_{\ d} + R^{a\gamma\omega d}h^{\rho}_{\ d} + R^{a\omega\rho d}h^{\gamma}_{\ d} + R^{a\omega\gamma d}h^{\rho}_{\ d}]\\
&+ B\partial_a\partial_\omega[-\eta^{\rho\omega}R^{ad}h^{\gamma}_{\ d} - \eta^{\gamma\omega}R^{ad}h^{\rho}_{\ d} + R^{a\omega}h^{\gamma\rho} + R^{\gamma\rho}h^{\omega a} - R^{\rho\omega}h^{\gamma a} - R^{\gamma\omega}h^{\rho a}]\\
&+ \frac{1}{2}B\partial_a\partial_\omega[-\eta^{a\rho}R^{\gamma d}h^{\omega}_{\ d} - \eta^{a\gamma}R^{\rho d}h^{\omega}_{\ d} + \eta^{a\omega}R^{\rho d}h^{\gamma}_{\ d} + \eta^{a\omega}R^{\gamma d}h^{\rho}_{\ d} + \eta^{a\rho}R^{\omega d}h^{\gamma}_{\ d} + \eta^{a\gamma}R^{\omega d}h^{\rho}_{\ d}]\\
&+ 2C\partial_a\partial_\omega[\eta^{a\omega}Rh^{\gamma\rho} - \eta^{\rho\omega}Rh^{\gamma a} - \eta^{\gamma\omega}Rh^{\rho a}]. \quad (3.35)
\end{aligned}
$$

It should be obvious at this point that there is no way we can reconcile this Hilbert energy-momentum tensor with what is uniquely derived from Noether's theorem for the linearized Gauss-Bonnet model, $T_N^{\omega\nu}$ in Equation (4.27), if we fix coefficients $A = \frac{1}{4}$, $B = -1$ and $C = \frac{1}{4}$ above. There is also a difference in corresponding coefficients between the terms the two share in common ($\eta^{\gamma\rho}[AR^{abcd}R_{abcd}+BR^{bd}R_{bd}+CR^2]-8AR^{\gamma bcd}R^{\rho}{}_{bcd}-B(4R^{\gamma d}R^{\rho}_{d}+4R^{\gamma b\rho d}R_{bd})-8CR^{\gamma\rho}R$), meaning even this part is not equivalent. To prove that this does not equal to the Noether energy-momentum tensor ($T_N^{\omega\nu} \neq T_H^{\gamma\rho}$) we simply can compare the part proportional to $\eta^{\gamma\rho}$. The rest we will abbreviate as $\mathcal{T}^{\gamma\rho}_{HNMP}$ to represent the Hilbert non-Minkowski part. This allows us to write the Hilbert energy-momentum tensor in compact form,

$$T_H^{\gamma\rho} = \eta^{\gamma\rho}\bigg(AR^{abcd}R_{abcd} + C(R^2 + 8\partial_\omega[R\bar{\Gamma}^{\omega b}{}_b] + 2\partial_a\partial_\omega[Rh^{\omega a}]) + B(R^{bd}R_{bd} + 4\partial_\omega[R^{bd}\bar{\Gamma}^{\omega}{}_{bd}] + \partial_a\partial_\omega[R^{ad}h^{\omega}{}_d])\bigg) + \mathcal{T}^{\gamma\rho}_{HNMP}. \quad (3.36)$$

Comparing the Noether and Hilbert energy-momentum tensors for linearized Gauss-Bonnet gravity

We will now fix coefficients of the Hilbert expression for the specific counter-example of linearized Gauss-Bonnet gravity. Setting $A = \frac{1}{4}$, $B = -1$ and $C = \frac{1}{4}$ yields,

$$T_H^{\gamma\rho} = \frac{1}{4}\eta^{\gamma\rho}\bigg(R^{abcd}R_{abcd} - 4R^{bd}R_{bd} + R^2 + 8\partial_\omega[R\bar{\Gamma}^{\omega b}{}_b] + 2\partial_a\partial_\omega[Rh^{\omega a}] - 16\partial_\omega[R^{bd}\bar{\Gamma}^{\omega}{}_{bd}] - 4\partial_a\partial_\omega[R^{ad}h^{\omega}{}_d]\bigg) + \mathcal{T}^{\gamma\rho}_{HNMP}. \quad (3.37)$$

Writing the Noether energy-momentum tensor in terms of the non-Minkowski part abbreviated as $\mathcal{T}^{\gamma\rho}_{NNMP} = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu}$,

$$T_N^{\omega\nu} = \frac{1}{4}\eta^{\omega\nu}(R_{\mu\lambda\alpha\beta}R^{\mu\lambda\alpha\beta} - 4R_{\mu\gamma}R^{\mu\gamma} + R^2) + \mathcal{T}^{\gamma\rho}_{NNMP}. \quad (3.38)$$

Subtracting the expressions for the Hilbert and Noether energy-momentum tensors given above yields,

$$T_H^{\gamma\rho} - T_N^{\omega\nu} = \frac{1}{4}\eta^{\gamma\rho}\bigg(8\partial_\omega[R\bar{\Gamma}^{\omega b}{}_b] + 2\partial_a\partial_\omega[Rh^{\omega a}] - 16\partial_\omega[R^{bd}\bar{\Gamma}^{\omega}{}_{bd}] - 4\partial_a\partial_\omega[R^{ad}h^{\omega}{}_d]\bigg) + \mathcal{T}^{\gamma\rho}_{HNMP} - \mathcal{T}^{\gamma\rho}_{NNMP}. \quad (3.39)$$

It is obvious $\mathcal{T}^{\gamma\rho}_{HNMP} - \mathcal{T}^{\gamma\rho}_{NNMP} \neq 0$. But to prove it for the $\eta^{\gamma\rho}$ part we will expand the derivatives,

$$T^{\gamma\rho}_{H} - T^{\omega\nu}_{N} = \frac{1}{4}\eta^{\gamma\rho}\Big((\eta^{bd}R - 2R^{bd})(-4\partial_\omega\partial^\omega h_{bd} + 10\partial_\omega\partial_d h^{\omega}_{\ b}) + (\eta^{bd}\partial_\omega R - 2\partial_\omega R^{bd})(-4\partial^\omega h_{bd} + 10\partial_d h^{\omega}_{\ b})\Big) + \mathcal{T}^{\gamma\rho}_{HNMP} - \mathcal{T}^{\gamma\rho}_{NNMP}. \quad (3.40)$$

None of the remaining terms cancel. Therefore we have proven that $T^{\omega\nu}_{N} \neq T^{\gamma\rho}_{H}$ in the case of linearized Gauss-Bonnet gravity. This completes the disproof by counterexample of the notion that the Hilbert and Noether energy-momentum tensors are, in general, equivalent.

3.1.5 Why the Noether and Hilbert energy-momentum tensors are not, in general, equivalent

We have now disproved by counterexample the conjecture that the Noether and Hilbert energy-momentum tensors are, in general, equivalent. This notion is often asserted based on the fact that for simple models such as electrodynamics, with first order derivatives of a vector potential in the Lagrangian, the two methods indeed yield the same result. Any proofs of equivalence have relied on assuming first order derivatives of first rank (vector) models [76, 170], as discussed in the Motivation section. Therefore we considered a model with second order derivatives of a second rank tensor potential in the Lagrangian (linearized higher order gravity models), and in particular the linearized Gauss-Bonnet gravity model which has a unique and well established energy-momentum tensor. In this case, the Noether and Hilbert energy-momentum tensors are not equivalent.

In this section we will conclude by explaining the three major reasons these two methods (which are very different mathematically) diverge from one another for higher order models and higher rank tensor potentials.

Reason 1: The Noether energy-momentum tensor $T^{\omega\nu}_{N}$ has terms proportional to $\eta^{\gamma\omega}$ which are exactly the Lagrangian density $\eta^{\gamma\omega}\mathcal{L}$ as seen in equation (4.20). This piece is always exactly proportional to the Lagrangian, which follows directly from application of Noether's first theorem. The Hilbert method, for models such as the linearized gravity models defined in equation (3.14), produces terms proportional to $\eta^{\gamma\omega}$ beyond what is present in the Lagrangian. In such cases, the Noether and Hilbert methods yield different results.

Reason 2: The covariant derivatives of higher rank tensor potentials do not cancel as in the case of simple models like the scalar field (Klein-Gordon) and vector field (electrodynamics).

For the covariant derivative of a scalar we have only $\nabla_\mu\phi = \partial_\mu\phi$, therefore we get no part with Christoffel symbols in the curved space Lagrangian. In electrodynamics, again we have $F^\nabla_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha - \partial_\nu A_\mu + \Gamma^\alpha_{\nu\mu}A_\alpha = F_{\mu\nu}$, where the $\Gamma^\alpha_{\mu\nu}$ contributions exactly cancel due to the antisymmetry of the field strength tensor, thus there is again no Christoffel part to the curved space Lagrangian. For models where the Christoffel symbols do not cancel, such as the linearized higher derivative gravity models given in equation (3.14) and considered in this paper, there are many lingering contributions from Christoffel symbols that simply add many terms which do not follow from Noether's method.

Reason 3: The proportional to Minkowski piece ($\eta^{\gamma\omega}\mathcal{L}$) derived from Noether's theorem in Equation (4.20) is recovered from the Hilbert method by differentiating the $\sqrt{-g}$ part of the curved space action. The remaining terms are built by contracting the expression with the metric. In electrodynamics this scalar is built from 2 metrics, for the linearized gravity model we consider there will be 4 metrics, and so on. The higher the rank of tensors one builds a Lagrangian from, the more metric contractions the curved space Lagrangian will have, but there will always be only one $\sqrt{-g}$ contribution. Therefore the relative contribution of the $\eta^{\gamma\omega}$ piece to the non $\eta^{\gamma\omega}$ piece in the Hilbert expression differs as we increase the order of derivatives and rank of fields, yet the $\eta^{\gamma\omega}\mathcal{L}$ piece from Noether has the same relationship to the non $\eta^{\gamma\omega}$ part regardless of the rank of tensor or order of derivatives. In simple cases such as a first order scalar field (Klein-Gordon), and first order vector field (electrodynamics) the proportion of these two contributions in the Hilbert method is identical to the Noether method. What this means is that even if we ignore problematic Christoffel terms from Reason 3, the Hilbert energy-momentum tensor would still not coincide with the Noether energy-momentum tensor.

These reasons provide some intuitive insight into why the Noether and Hilbert energy-momentum tensors are not in general giving the same result, without need to consider the details of the more technical disproof provided in this article. The obvious question that now arises is, if we have two methods for deriving energy-momentum tensors from a Lagrangian density of a model in Minkowski spacetime, and they are not generally giving equivalent results, which one should be considered like the fundamental method for deriving physical expressions? In the case of equations of motion derived from a Lagrangian density, the Euler-Lagrange equation has no such 'different' method to ' compete' with. The connection of the Noether method to the Euler-Lagrange equation, coupled with its connection to symmetry and to the derivation of the unique and well accepted expression for linearized Gauss-Bonnet gravity used in the disproof by counterexample in this article, seem to speak for itself. Any more general, philosophical thoughts related to this decision will be left to the reader.

3.2 Energy-momentum tensor variants of canonical Noether, Hilbert, Belinfante, Fock and 'new improved' for the general Lagrangian density of Klein-Gordon theory

Abstract In the literature, multiple methods for deriving the energy-momentum tensor of a physical field theory exist. Recent research proved that the physical energy-momentum tensor derived directly from Noether's first theorem is not generally equivalent to the Hilbert (metric) energy-momentum tensor in Minkowski spacetime. Some of the more common expressions include the canonical-Noether, Hilbert, Belinfante and Fock energy-momentum tensors. While these methods are mathematically distinct, it is often asserted that they are in some sense generally equivalent; this sentiment is due to the equivalence of multiple methods for particular models such as conventional Klein-Gordon and electrodynamics. In this letter, we consider the most general Lagrangian density leading to the Klein-Gordon equation of motion. We show for different choices of free coefficients in this most general Lagrangian density, vastly different conclusions regarding the 5 standard energy-momentum tensor definitions of a scalar field can be made. Using this general Lagrangian density we prove several new results regarding the relationship between the various energy-momentum tensor definitions, such as that the Hilbert and Belinfante expressions cannot in general be related even on-shell, and that an off-shell traceless energy-momentum tensor exists for Klein-Gordon theory. This reinforces the recent results in the literature, and suggests while in simple cases these definitions can all coincide, that this is merely a coincidence and not a general feature of the various mathematical approaches. Due to this fact we argue that a unique energy-momentum tensor definition should be adopted to avoid ambiguities caused by these contradictions. In light of this letter and other recent results, we argue in favour of the Noether approach.

3.2.1 Motivation

The energy-momentum tensor of a physical field theory is an object of fundamental significance. In electrodynamics, for example, the energy-momentum tensor compactly expresses the familiar Lorentz force law and Poynting's theorem. Deriving the equation of motion for a physical theory is straightforward; the Euler-Lagrange equation is unique for a particular theory. The energy-momentum tensor on the other hand is not so straightforward - there exists many different methods for deriving an energy-momentum tensor for a given model. Recent research has proven the two most common methods, namely Noether's first theorem and the Hilbert (metric) method, do not generally yield the same expression for special relativistic field theories in Minkowski spacetime [13]. We note that this result corresponds to the physical energy-momentum tensor [29] (symmetric, gauge invariant, conserved and trace-free) derived directly from Noether's first theorem using the Bessel-Hagen method [26] described in section 2.2.2. of [29]. This result does not correspond to the so-called canonical Noether energy-momentum tensor tensor which is trivially not equivalent to the symmetric Hilbert expression (except for the Klein-Gordon scalar field theory); the canonical Noether tensor is what follows from Noether's first theorem considering only the 4-parameter Poincare translation. Since many definitions for the energy-momentum tensor exist, and they do not generally coincide, it is a question of fundamental significance as to which consistently derived the unique physical energy-momentum tensor for a given theory.

Four of the common energy-momentum tensors in Minkowski spacetime are the (i) canonical Noether, (ii) Hilbert, (iii) Belinfante and (iv) Fock expressions [29, 75]. For electrodynamics, (ii)-(iv) are identical, which contributed to the impression that these very different mathematical approaches are in some sense generally equivalent. Attempted proofs of the equivalence between (ii) and (iii) have a long history [95, 76, 170], with the best attempts concluding that the two expressions can differ by combinations of the equations of motion [170]. The exception to this rule is for scalar fields, where the conventional Klein-Gordon model has equivalence for all (i)-(iv) [168]. This is because the spin angular momentum for a scalar field is identically zero, thus (i) and (iii) are identical. The conventional Klein-Gordon (KG) model [121, 93] (although supposedly first developed by Schrodinger [191]) has a well known Lagrangian density $\mathcal{L}_{KG}$, equation of motion E_{KG}, and energy-momentum tensor $T^{\gamma\rho}_{KG}$,

$$\mathcal{L}_{KG} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{3.41}$$

$$E_{KG} = \Box\phi, \tag{3.42}$$

$$T^{\gamma\rho}_{KG} = -\frac{1}{2}\eta^{\gamma\rho}\partial_\mu\phi\partial^\mu\phi + \partial^\gamma\phi\partial^\rho\phi. \tag{3.43}$$

The question remains however, for the most general scalar field model (with free coefficients in the Lagrangian density given in (3.44)), are there particular Lagrangians that differentiate the various methods (i)-(iv)? For example, does a particular scalar field model exist such that (ii) and (iii) cannot be reconciled, since for scalar field model the Belinfante improvement has identically zero spin connections? Given the fundamental significance of an energy-momentum tensor for a physical theory, cases where (i)-(iv) cannot be reconciled in some way are of high importance; if there are several energy-momentum tensors for a given model, one must be selected to write down the set of fundamental conservation laws (as in the case of electrodynamics). In this letter we will consider a general linear combination of the scalar field Lagrangian density (3.44) that yields the Klein-Gordon equation (3.45) in order to elucidate that even for a simple scalar field theory, different choices of the parameters A and B can lead to vastly different equivalence/ non-equivalence of the various energy-momentum tensors. To explore these questions we will consider the most general Klein-Gordon Lagrangian density,

$$\mathcal{L} = A\partial_\mu\phi\partial^\mu\phi + B\phi\partial_\mu\partial^\mu\phi, \tag{3.44}$$

whose Euler derivative $E = \frac{\delta\mathcal{L}}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} + \partial_\alpha\partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)}$ is the ordinary Klein-Gordon equation of motion,

$$E = (2B - 2A)\Box\phi, \tag{3.45}$$

and apply methods (i)-(iv) to this Lagrangian density. Focusing on the general scalar model allows for simple calculations/ comparisons of the various methods (i)-(iv), whose results we summarize in section 4. The general Lagrangian density (3.44) allows one to explore the system of linear coefficients that result from each of the energy-momentum tensors (given in section 2) in order to see for what solutions the results can be reconciled, and for what solutions the results obtain known results in the literature. In this sense (3.44) and (3.45) are a toy model which we use to elucidate problems with having a variety of energy-momentum tensor definitions.

In addition we will consider a fifth expression, 'a new improved energy-momentum tensor' of Callan-Coleman-Jackiw [45]. Since (3.43) is not traceless (on or off-shell), the authors [45] proposed an ad-hoc improvement to obtain an on-shell tracefree expression; we show that methods (i), (iii) and (iv) can be used to derive the 'new improved energy-momentum tensor' directly from (3.44) without any improvements needed. This result was to some extent noted in the past by Kuzmin and McKeon in [127]. Our notable results are summarized in section 4.

3.2.2 Various energy-momentum tensor definitions

We will now calculate the various energy-momentum tensors for the general Klein-Gordon Lagrangian (3.44), which must be conserved on-shell up to (3.45).

(i) canonical Noether energy-momentum tensor

From Noether's first theorem for a scalar field ϕ Lagrangian (3.44) we can derive Noether's differential identity between the equation of motion and conservation law,

$$\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} + \partial_\alpha \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\beta \phi)}\right)\delta\phi$$
$$+ \partial_\mu \left(\eta^{\mu\nu}\mathcal{L}\delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \phi)}\partial_\omega \delta\phi - \left[\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \phi)}\right]\delta\phi\right) = 0\,. \quad (3.46)$$

This identity relates the equation of motion E (first line above) to the Noether current j^μ (second line above) as $E\delta\phi + \partial_\mu j^\mu = 0$. The canonical Noether energy-momentum tensor is defined as the Noether current j^μ when the transformation of fields is defined as the 4-parameter Poincare translation $\delta x_\nu = a_\nu$ and transformation of fields $\delta\phi = -(\partial^\nu\phi)\delta x_\nu$. Substituting these into (3.46) and factoring out the constant a_ν from j^μ we are left with the canonical Noether energy-momentum tensor $T_C^{\mu\nu}$,

$$T_C^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \phi)}\partial_\omega \partial^\nu\phi + \left[\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \phi)}\right]\partial^\nu\phi. \quad (3.47)$$

Inserting (3.44) into (3.47) we obtain,

$$T_C^{\mu\nu} = \eta^{\mu\nu}(A\partial_\alpha\phi\partial^\alpha\phi + B\phi\partial_\alpha\partial^\alpha\phi) + (B - 2A)\partial^\mu\phi\partial^\nu\phi - B\phi\partial^\mu\partial^\nu\phi. \quad (3.48)$$

This is the most general canonical Noether energy-momentum tensor for Klein-Gordon theory. In section 3 we will compare this expression to the other energy-momentum tensor definitions for various solutions A and B.

(ii) Hilbert (metric) energy-momentum tensor in Minkowski spacetime

The Hilbert (metric) energy-momentum tensor in Minkowski spacetime [29] is defined as,

$$T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_H}{\delta g_{\gamma\rho}}\bigg|_{g=\eta}. \quad (3.49)$$

This method involves writing a special relativistic Lagrangian in 'curved space' by replacing all derivatives with covariant derivatives $\partial \to \nabla$, replacing all Minkowski metrics with the metric tensor $\eta \to g$ and introducing the Jacobian term $\sqrt{-g}$. Writing (3.44) in 'curved space' we have,

$$\mathcal{L}_H = \sqrt{-g}g^{\mu\nu}(A\nabla_\mu\phi\nabla_\nu\phi + B\phi\nabla_\mu\nabla_\nu\phi). \quad (3.50)$$

For a scalar field the 'curved space' Lagrangian $\mathcal{L}_H$ is simplified because the covariant derivative of a scalar is trivially $\nabla_\mu\phi = \partial_\mu\phi$. Therefore we are left with a single covariant derivative in the second term $\mathcal{L}_H = \sqrt{-g}g^{\mu\nu}(A\partial_\mu\phi\partial_\nu\phi + B\phi\nabla_\mu\partial_\nu\phi)$. This covariant derivative is $\nabla_\mu\partial_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi$, where the Christoffel symbol of the second kind is $\Gamma^\lambda_{\nu\alpha} = \frac{1}{2}g^{m\lambda}(-\partial_m g_{\nu\alpha} + \partial_\alpha g_{m\nu} + \partial_\nu g_{m\alpha})$. Thus we have for (3.50),

$$\mathcal{L}_H = \sqrt{-g}g^{\mu\nu}(A\partial_\mu\phi\partial_\nu\phi + B\phi\partial_\mu\partial_\nu\phi - B\phi\Gamma^\lambda_{\mu\nu}\partial_\lambda\phi). \tag{3.51}$$

We now require the Euler derivative $\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}} = \frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} - \partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})}$ for (3.49). For this we will need $\frac{\partial\sqrt{-g}}{\partial g_{\gamma\rho}} = \frac{1}{2}g^{\gamma\rho}\sqrt{-g}$ and $\frac{\partial g^{\mu\nu}}{\partial g_{\gamma\rho}} = -\frac{1}{2}(g^{\gamma\mu}g^{\rho\nu} + g^{\rho\mu}g^{\gamma\nu})$. Calculating the required derivatives,

$$\frac{\partial\mathcal{L}}{\partial g_{\gamma\rho}} = \frac{1}{2}\sqrt{-g}(A\partial_\mu\phi\partial_\nu\phi + B\phi\partial_\mu\partial_\nu\phi)[g^{\mu\nu}g^{\gamma\rho} - g^{\gamma\mu}g^{\rho\nu} - g^{\rho\mu}g^{\gamma\nu}], \tag{3.52}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\omega g_{\gamma\rho})} = -\frac{1}{2}B(-g^{\omega\lambda}g^{\gamma\rho} + g^{\gamma\lambda}g^{\rho\omega} + g^{\rho\lambda}g^{\omega\gamma})\sqrt{-g}\phi\partial_\lambda\phi, \tag{3.53}$$

and inserting into (3.49) and returning to Minkowski space we have,

$$\begin{aligned} T^{\gamma\rho}_H &= (A\partial_\mu\phi\partial_\nu\phi + B\phi\partial_\mu\partial_\nu\phi)[\eta^{\mu\nu}\eta^{\gamma\rho} - \eta^{\gamma\mu}\eta^{\rho\nu} - \eta^{\rho\mu}\eta^{\gamma\nu}] \\ &\quad + B(-\eta^{\omega\lambda}\eta^{\gamma\rho} + \eta^{\gamma\lambda}\eta^{\rho\omega} + \eta^{\rho\lambda}\eta^{\omega\gamma})\partial_\omega\phi\partial_\lambda\phi + B(-\eta^{\omega\lambda}\eta^{\gamma\rho} + \eta^{\gamma\lambda}\eta^{\rho\omega} + \eta^{\rho\lambda}\eta^{\omega\gamma})\phi\partial_\omega\partial_\lambda\phi. \end{aligned} \tag{3.54}$$

Contracting all Minkowski metrics and collecting like terms we are left with the Hilbert (metric) energy-momentum tensor in Minkowski spacetime,

$$T^{\gamma\rho}_H = (A - B)\eta^{\gamma\rho}\partial_\mu\phi\partial^\mu\phi + (2B - 2A)\partial^\gamma\phi\partial^\rho\phi. \tag{3.55}$$

We note that this expression is conserved for all A, B since $\partial_\gamma T^{\gamma\rho}_H = (2A - 2B)[\partial_\gamma\phi\partial^\rho\partial^\gamma\phi - \partial_\gamma\phi\partial^\gamma\partial^\rho\phi] = 0$. This is the most general Hilbert energy-momentum tensor for Klein-Gordon theory. In section 3 we will compare this expression to the other energy-momentum tensor definitions for various solutions A and B.

(iii) Belinfante energy-momentum tensor

The Belinfante energy-momentum tensor is defined as the canonical Noether energy-momentum tensor plus a specific improvement term [23]. This 'symmetrization procedure' is designed to add a specific divergence of a superpotential $\partial_\alpha b^{[\mu\alpha]\nu}$ that will eliminate the antisymmetric part of $T^{\mu\nu}_C$,

$$T^{\mu\nu}_B = T^{\mu\nu}_C + \partial_\alpha b^{[\mu\alpha]\nu}, \tag{3.56}$$

since $T_C^{\mu\nu}$ is not automatically symmetric. However, in the case of the general scalar field Lagrangian (3.44), we automatically have a symmetric canonical Noether energy-momentum tensor in (3.48); hence the Belinfante symmetrization improvement has no sense in this case. This is well justified because the Belinfante superpotential is based on the spin connection $S^{\rho[\sigma\gamma]}$ of the model,

$$b^{[\rho\gamma]\sigma} = \frac{1}{2}(-S^{\rho[\sigma\gamma]} + S^{\gamma[\sigma\rho]} + S^{\sigma[\gamma\rho]}), \tag{3.57}$$

where $b^{[\rho\gamma]\sigma}$ is antisymmetric in $[\rho\gamma]$ so that conservation $\partial_\rho\partial_\gamma b^{[\rho\gamma]\sigma} = 0$ is independent and does not affect the already conserved canonical Noether tensor. We note that in [23], Belinfante refers to an uncited *'Dr. Podolansky'* who suggested this result to him. The spin connection $S^{\rho[\sigma\gamma]}$ for a scalar field theory is identically zero (it has no such contribution), thus the superpotential $b^{[\rho\gamma]\sigma} = 0$ and we have the expected result for a scalar field that [168],

$$T_B^{\mu\nu} = T_C^{\mu\nu}. \tag{3.58}$$

This relationship is important to emphasize because many claims of the relationship between the Belinfante and Hilbert energy-momentum tensors exist [95, 76, 170]. However in the case of scalar field theory we can directly compare the canonical Noether result to the Hilbert result without the need for ad-hoc improvement; the Belinfante tensor is simply the canonical Noether tensor in this case. The need for ad-hoc improvement of energy-momentum tensors is often criticized, as discussed by Forger and Römer [76]: *There is a long history of attempts to cure these diseases and arrive at the physically correct energy-momentum tensor $T_{\mu\nu}$ by adding judiciously chosen "improvement" terms to $T_C^{\mu\nu}$*. They go on to say: *However, all these methods of defining improved energy-momentum tensors are largely "ad hoc" prescriptions focussed on special models of field theory, often geared to the needs of quantum field theory and ungeometric in spirit*. Studying the various definitions for a scalar field theory allows us to largely avoid these ad-hoc manipulations and focus on possible results of the direct and unique calculations.

(iv) Fock energy-momentum tensor

The Fock energy-momentum tensor [75] is defined as the most general linear combination of second rank symmetric terms for a given theory, which is solved for possible on-shell conserved expressions. This gives the most general picture of possible conserved symmetric energy-momentum tensors, irrespective of the various other methods. For (3.44) we have,

$$T_F^{\gamma\rho} = \eta^{\gamma\rho}(C\partial_\alpha\phi\partial^\alpha\phi + D\phi\partial_\alpha\partial^\alpha\phi) + E\partial^\gamma\phi\partial^\rho\phi + F\phi\partial^\gamma\partial^\rho\phi. \tag{3.59}$$

Taking the divergence of this expression,

$$\partial_\gamma T_F^{\gamma\rho} = (2C + E + F)\partial_\alpha\phi\partial^\alpha\partial^\rho\phi + (D + E)\partial^\rho\phi\partial_\alpha\partial^\alpha\phi + (D + F)\phi\partial_\alpha\partial^\alpha\partial^\rho\phi. \qquad (3.60)$$

We have the condition $2C + E + F = 0$ which must hold in (3.59) for it to be conserved on-shell. This is the most general Fock energy-momentum tensor for Klein-Gordon theory. In section 3 we will compare this expression to the other energy-momentum tensor definitions for various solutions C, D, E and F.

(v) 'A new improved energy-momentum tensor' by Callan-Coleman-Jackiw

The conventional Klein-Gordon energy-momentum tensor (3.43) is not traceless. The tracelessness of the energy-momentum tensor is an essential characteristic of a physical field theory [29]. This was noted by Callan-Coleman-Jackiw in 1970 [45], who proposed 'A new improved energy-momentum tensor' that improved the canonical Noether tensor $T_C^{\mu\nu}$ of ordinary Klein-Gordon theory; when $A = -\frac{1}{2}$ and $B = 0$ in (3.44). They presented,

$$T_{CCJ}^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \partial^\mu\phi\partial^\nu\phi - \frac{1}{6}(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial_\alpha\partial^\alpha)\phi^2, \qquad (3.61)$$

where the first two terms are $T_C^{\mu\nu}$ for $A = -\frac{1}{2}$ and $B = 0$, and the final two terms are their 'new improvement'. This improvement was defined with a very specific purpose; to have an on-shell tracelessness of the energy-momentum tensor. This expression can be expanded straightforwardly for easier comparison to the results in the previous sections,

$$T_{CCJ}^{\mu\nu} = -\frac{1}{6}\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{3}\eta^{\mu\nu}\phi\partial_\alpha\partial^\alpha\phi + \frac{2}{3}\partial^\mu\phi\partial^\nu\phi - \frac{1}{3}\phi\partial^\mu\partial^\nu\phi. \qquad (3.62)$$

This 'new improved energy-momentum tensor' is the ad-hoc improved energy-momentum tensor derived in [45]. In section 3 we will compare it to the other energy-momentum tensor definitions.

3.2.3 Solving the free coefficients for desired properties of the energy-momentum tensors

Now that we have definitions (i)-(iv) in terms of free coefficients, and a fifth definition (v), we can consider solutions that will reconcile or differentiate the different methods.

Recovering the conventional Klein-Gordon energy-momentum tensor from definitions (i)-(iv)

We start by noting the trival result, that for conventional Klein-Gordon theory, where (3.44) has $B = 0$,

$$\mathcal{L} = A\partial_\mu\phi\partial^\mu\phi. \tag{3.63}$$

Methods (i)-(iii) are identical for any choice of the free parameter A,

$$T_C^{\gamma\rho} = T_H^{\gamma\rho} = T_B^{\gamma\rho} = A(\eta^{\gamma\rho}\partial_\mu\phi\partial^\mu\phi - 2\partial^\gamma\phi\partial^\rho\phi). \tag{3.64}$$

This is the conventional energy-momentum tensor for Klein-Gordon theory [168], typically presented with $A = \pm\frac{1}{2}$. In addition the Fock method $T_F^{\gamma\rho}$ in (3.59) trivially recovers this energy-momentum tensor for $C = A$, $D = 0$, $E = -2A$ and $F = 0$, as it satisfies $2C + E + F = 0$. Thus, as is well known, methods (i)-(iv) can all be reconciled for the standard $B = 0$ Lagrangian.

Deriving 'a new improved energy-momentum tensor' directly from Noether's first theorem

The result (3.62) in [45] implies that ad-hoc improvements are needed to obtain (3.62) from Noether's theorem. The improvement of energy-momentum tensors is of course undesirable as discussed by Forger and Römer [76]. It is far superior for an energy-momentum tensor to follow directly from a well defined method such as Noether's first theorem or the Hilbert (metric) energy-momentum tensor without any manipulations required. This is exactly the case for a particular choice of A and B in the general Lagrangian (3.44). For the solution $A = -\frac{1}{6}$ and $B = \frac{1}{3}$ we have the result,

$$T_C^{\mu\nu} = T_B^{\mu\nu} = T_{CCJ}^{\mu\nu} = -\frac{1}{6}\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{3}\eta^{\mu\nu}\phi\partial_\alpha\partial^\alpha\phi + \frac{2}{3}\partial^\mu\phi\partial^\nu\phi - \frac{1}{3}\phi\partial^\mu\partial^\nu\phi. \tag{3.65}$$

To some extent this result was realized by Kuzmin and McKeon [127] via the additional surface term considered for the conventional Klein-Gordon action, that can be added to obtain $T_{CCJ}^{\mu\nu}$ directly from Noether's first theorem. In addition the Fock method $T_F^{\gamma\rho}$ in (3.59) trivially recovers this energy-momentum tensor for $C = -\frac{1}{6}$, $D = \frac{1}{3}$, $E = \frac{2}{3}$ and $F = -\frac{1}{3}$, as it satisfies $2C + E + F = 0$. The Hilbert energy-momentum tensor (3.55) can of course not recover this for any solution A and B. Note that $A = -\frac{1}{6}$ and $B = \frac{1}{3}$ yields $T_H^{\gamma\rho} = -\frac{1}{2}\eta^{\gamma\rho}\partial_\mu\phi\partial^\mu\phi + \partial^\gamma\phi\partial^\rho\phi$ which is simply the conventional (3.64) for $A = -\frac{1}{2}$ and $B = 0$. Therefore methods (i) and (iii)-(v) can be reconciled as the 'new improved energy-momentum tensor'.

On-shell tracelessness conditions of the various energy-momentum tensors (i)-(iv)

The 'new improved energy-momentum tensor' in (3.62) is on-shell traceless by construction. The conventional energy-momentum tensor for Klein-Gordon theory (3.64) is not. We can ask the question for the general tensors (i)-(iii) be solved for tracelessness in a more general manner. From (3.48) we have,

$$\eta_{\gamma\rho}T_C^{\gamma\rho} = (2A + B)\partial_\alpha\phi\partial^\alpha\phi + 3B\phi\partial_\alpha\partial^\alpha\phi, \tag{3.66}$$

thus on-shell $3B\phi\partial_\alpha\partial^\alpha\phi = 0$ and we have the condition $2A+B = 0$ for on-shell tracelessness. Recall to derive $T^{\mu\nu}_{CCJ}$ we required $A = -\frac{1}{6}$ and $B = \frac{1}{3}$, which indeed satisfies this on-shell condition. Thus for all $B = -2A$ both of the (equivalent) (i) canonical Noether $T_C^{\gamma\rho}$ and (iii) Belinfante $T_B^{\gamma\rho}$ energy-momentum tensors are,

$$T_C^{\mu\nu} = T_B^{\gamma\rho} = A[\eta^{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi - 2\phi\partial_\alpha\partial^\alpha\phi) - 4\partial^\mu\phi\partial^\nu\phi + 2\phi\partial^\mu\partial^\nu\phi], \tag{3.67}$$

where we have an on-shell traceless result for all A (and of course $T^{\mu\nu}_{CCJ}$ in (3.62) is recovered for $A = -\frac{1}{6}$). For (ii) the Hilbert (metric) energy-momentum tensor in (3.55) on the other hand,

$$\eta_{\gamma\rho}T_H^{\gamma\rho} = (2A - 2B)\partial_\mu\phi\partial^\mu\phi. \tag{3.68}$$

We have tracelessness only for $A = B$. But (3.55) is identically zero ($T_H^{\gamma\rho} = 0$) for all $A = B$. Therefore there exists no non-trivial tracefree Hilbert energy-momentum tensor for (3.44). Finally for the Fock energy-momentum tensor (3.59) we have,

$$\eta_{\gamma\rho}T_F^{\gamma\rho} = (4C + E)\partial_\alpha\phi\partial^\alpha\phi + (4D + F)\phi\partial_\alpha\partial^\alpha\phi, \tag{3.69}$$

thus on-shell $(4D + F)\phi\partial_\alpha\partial^\alpha\phi = 0$ and we have the condition $4C + E = 0$ for on-shell tracelessness. Recall to derive $T^{\mu\nu}_{CCJ}$ we required $C = -\frac{1}{6}$ and $E = \frac{2}{3}$, which indeed satisfies this on-shell condition. Thus for all $E = -4C$ we have an on-shell traceless (iv) Fock energy-momentum tensor in (3.59),

$$T_F^{\gamma\rho} = \eta^{\gamma\rho}(C\partial_\alpha\phi\partial^\alpha\phi + D\phi\partial_\alpha\partial^\alpha\phi) - 4C\partial^\gamma\phi\partial^\rho\phi + F\phi\partial^\gamma\partial^\rho\phi, \tag{3.70}$$

which is tracefree on-shell for all C, D and F. Note that it recovers $T^{\mu\nu}_{CCJ}$ in (3.62) for $C = -\frac{1}{6}$, $D = \frac{1}{3}$ and $F = -\frac{1}{3}$.

Off-shell tracelessness conditions for (iv) the Fock energy-momentum tensor

The previous subsection explored the many possible on-shell tracelessness conditions for the various energy-momentum tensors (i)-(v). The downside of these results is that the on-shell condition must be imposed in order to obtain tracelessness. In classical electrodynamics, for example, $T^{\mu\nu}$ is tracefree off-shell, a stronger and more desirable result for the tracelessness of the physical expression.

We do note a result unique to the Fock energy-momentum tensor traceless conditions in (3.69); it is possible to construct an off-shell tracefree energy-momentum tensor for the scalar field as in the case of electrodynamics. For this expression, it is not necessary to use the on-shell condition: the energy-momentum tensor can be made traceless off-shell by fixing the coefficients as $4C + E = 0$ and $4D + F = 0$, yielding,

$$T_F^{\gamma\rho} = \eta^{\gamma\rho}(C\partial_\alpha\phi\partial^\alpha\phi + D\phi\partial_\alpha\partial^\alpha\phi) - 4C\partial^\gamma\phi\partial^\rho\phi - 4D\phi\partial^\gamma\partial^\rho\phi, \tag{3.71}$$

which is tracefree off-shell for all C, D. However this can be further simplified by the required conservation condition $2C + E + F = 0$, which for $E = -4C$ and $F = -4D$ yields the condition $C = -2D$. Imposing the conservation condition,

$$T_F^{\gamma\rho} = D[\eta^{\gamma\rho}(-2\partial_\alpha\phi\partial^\alpha\phi + \phi\partial_\alpha\partial^\alpha\phi) + 8\partial^\gamma\phi\partial^\rho\phi - 4\phi\partial^\gamma\partial^\rho\phi], \tag{3.72}$$

we have an off-shell tracefree, conserved, symmetric energy-momentum tensor as in the case of electrodynamic theory.

The (ii) Hilbert (metric) energy-momentum tensor can have less terms than the (i) canonical Noether energy-momentum tensor

The conventional view is that the canonical Noether energy-momentum tensor $T_C^{\mu\nu}$ must be improved (by adding additional terms) to obtain the physical energy-momentum tensor [29], typically by the addition of the Belinfante term in (3.103) [95, 76, 170]. The Hilbert energy-momentum tensor $T_H^{\mu\nu}$ for electrodynamics gives this physical expression immediately. The canonical Noether tensor is generally viewed to have less terms than the Hilbert energy-momentum tensor, which leads to the conventional wisdom that terms must be supplemented ad-hoc to $T_C^{\mu\nu}$ to reconcile it with $T_H^{\mu\nu}$. In addition it is often claimed that the Belinfante term is at least part of this reconciliation [95, 76, 170]. Here we give a counterexample to this general view.

In the case of (3.44), the canonical Noether energy-momentum tensor (3.48) has more terms than the Hilbert energy-momentum tensor (3.55). Based on the conventional wisdom we should be adding terms to (3.48) to obtain (3.55). Instead (3.48) has the additional terms,

$$T_C^{\mu\nu} - T_H^{\mu\nu} = B[\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \eta^{\mu\nu}\phi\partial_\alpha\partial^\alpha\phi - \partial^\mu\phi\partial^\nu\phi - \phi\partial^\mu\partial^\nu\phi]. \tag{3.73}$$

Note that the additional terms in $T_C^{\mu\nu}$ all depend on B, which explains why in the conventional Klein-Gordon theory for $B = 0$ these results all reconcile, as shown in (3.64).

The (ii) Hilbert (metric) energy-momentum tensor and the (iii) Belinfante energy-momentum tensor are not equivalent, even on-shell

The previous results attempt to claim that the (ii) Hilbert (metric) energy-momentum tensor and the (iii) Belinfante energy-momentum tensor are in some sense generally equivalent, at least using on-shell conditions [95, 76, 170]. For these scalar models the (iii) Belinfante energy-momentum tensor is equivalent to (i) the canonical Noether energy-momentum tensor, thus from (3.58) and (3.73) we have,

$$T_B^{\mu\nu} - T_H^{\mu\nu} = B[\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \eta^{\mu\nu}\phi\partial_\alpha\partial^\alpha\phi - \partial^\mu\phi\partial^\nu\phi - \phi\partial^\mu\partial^\nu\phi]. \tag{3.74}$$

Even if we use the on-shell condition we have the result $T_B^{\mu\nu} - T_H^{\mu\nu} \neq 0$. Therefore we have proven by counterexample that (ii) and (iii) need not be related, even using on-shell conditions.

3.2.4 Summary of results

We now summarize the main results of the previous section, where the 5 energy-momentum tensors we refer to are (i) canonical Noether, (ii) Hilbert (metric), (iii) Belinfante, (iv) Fock and (v) 'new improved' of Callan-Coleman-Jackiw [45].

- From (3.44) the general energy-momentum tensors (i)-(iii) are identical to the conventional Klein-Gordon energy-momentum tensor (3.64) when $B = 0$, for all A. This expression can be recovered from (iv), but is not equivalent to (v) for any A.

- Energy-momentum tensor (v) can be derived directly from Noether's first theorem for (i) and (iii) in (3.65) without the need for ad-hoc improvement terms. To some extent this result was realized in the past in [127], by considering a surface term added to the conventional Klein-Gordon Lagrangian. We note (v) was also obtained from (iv).

- On-shell tracelessness of (i) and (iii) was obtained in (3.67) for a general parameter A. There exists no non-trivial (ii) which is traceless on-shell or off-shell. A more general on-shell traceless energy-momentum tensor (3.70) was found for (iv).

- Off-shell tracelessness is not possible for (i)-(iii) and (v). However we obtained for (iv) the energy-momentum tensor (3.72) which is traceless off-shell, conserved and symmetric; all properties of the unique physical energy-momentum tensors common in i.e. electrodynamics.

- Contrary to conventional wisdom, there exists for (3.44) for all $B \neq 0$ an energy-momentum tensor (i) that has more terms than (ii); the common view in the literature is that one must ad-hoc add terms to (i) in order to reconcile with (ii) since (i) is usually lacking terms found in (ii).

- Contrary to conventional wisdom that (ii) and (iii) are on-shell equivalent in general, for (3.44) these two tensors cannot be reconciled, not even on-shell.

3.2.5 Discussion

We have shown that from the most general possible Klein-Gordon Lagrangian density with free coefficients (3.44) and applying various methods of energy-momentum tensor derivation, five common definitions for scalar field theory (i) canonical Noether, (ii) Hilbert (metric), (iii) Belinfante, (iv) Fock and (v) new improved of Callan-Coleman-Jackiw can vary drastically based on the choice of the free coefficients. The method of using a general Lagrangian density with free coefficients was applied in [14] to derive unique gauge invariant energy-momentum tensors from Noether's first theorem. Considering a general Lagrangian density with free coefficients is a powerful method for answering questions about generality and uniqueness of equations in a particular theory.

The energy-momentum tensors (i)-(iv) for the standard Klein-Gordon Lagrangian (3.41) are well known to be equivalent [168] which is one of the reasons for the conventional wisdom of calling the mathematically distinct methods by the same name (energy-momentum tensor) and symbol ($T^{\mu\nu}$). However recent research proved that this equivalence does not generally hold for the physical energy-momentum tensors derived from Noether's first theorem and the Hilbert (metric) method in Minkowski spacetime [13]. This motivated us to consider the most general Lagrangian with free coefficients as in [14], and apply the various methods that exist in the literature. The fact that many of the energy-momentum tensors depend explicitly on the Lagrangian density requires a unique Lagrangian density for each given conservation law, contrary to the Euler-Lagrange equation where numerous different Lagrangains can yield the same result; this allowed us to explore the contradictions in different energy-momentum tensor definitions based on all possible (3.44) that give the same Klein-Gordon equation of motion

(3.45). As our summarized results in section 4 indicate, equivalence of (i)-(v) for a scalar field varies drastically based on the particular choice of free coefficients. Most notably we show that: (a) contrary to popular belief, there exists a Hilbert energy-momentum tensor with less terms than the Belinfante tensor that cannot be reconciled on-shell, (b) the 'new improved energy-momentum tensor' of Callan-Coleman-Jackiw can be derived directly from Noether's first theorem without any ad-hoc improvements needed, (c) from (iv) an off-shell trace-free energy-momentum tensor is possible and (d) no trace-free (on or off-shell) energy-momentum tensor can be obtained from the Hilbert method.

The wide variety of conclusions that can be made for such a simple model as the general Lagrangian density (3.44) for Klein-Gordon scalar field theory emphasizes the ambiguity problem in having multiple different mathematical definitions for something that is supposedly the same physical entity called an 'energy-momentum tensor'. This leads to several problems when writing down unique expressions for a given theory, such as conservation and force laws, where in electrodynamics i.e. Poynting's theorem and the Lorentz force law are uniquely defined by the uniquely accepted physical energy-momentum tensor for the theory. In cases such as spin-2, ambiguity problems arise from various distinct definitions of the energy-momentum tensor that alter fundamental calculations based on which $T^{\mu\nu}$ is selected, such as the controversy around the spin-2 self coupling $h_{\mu\nu}T^{\mu\nu}$ in attempted spin-2 to general relativity derivations [163]. Sorting out these ambiguities is one major reason why from the various energy-momentum tensors that exist in the literature, we as physicists should determine which is generally applicable to physical energy-momentum tensors (such as the universal application of the Euler-Lagrange equation of motion for a given model), and which methods happen to yield the same energy-momentum tensor by coincidence for simple models. Based on the inherent connection to the Euler-Lagrange equation and symmetries of the action, as emphasized by [13], we argue that it is the results of Noether's first theorem when the physical energy-momentum tensor [29] can be uniquely derived without any ad-hoc improvements [26] that should be considered as the unique and universal definition for energy-momentum derivation. However, arguments for the other methods should of course be taken into consideration as this is a foundationally important scientific problem at the heart of theoretical physics.

3.3 Canonical Noether and the energy-momentum non-uniqueness problem in linearized gravity

Abstract Recent research has highlighted the non-uniqueness problem of energy-momentum tensors in linearized gravity; many different tensors are published in the literature, yet for particular calculations a unique expression is required. It has been shown that (A) none of these spin-2 energy-momentum tensors are gauge invariant and (B) the Noether and Hilbert energy-momentum tensors are not, in general, equivalent; therefore uniqueness criteria is difficult to specify. Conventional wisdom states that the various published spin-2 energy-momentum tensors can be derived from the canonical Noether energy-momentum tensor by adding ad-hoc "improvement" terms (the divergence of a superpotential and terms proportional to the equations of motion), that these superpotentials are in some way unique or physically significant, and that this implies some meaningful connection to the Noether procedure. To explore this question of uniqueness, we consider the most general possible spin-2 energy-momentum tensor with free coefficients using the Fock method. We express this most general energy-momentum tensor as the canonical Noether tensor, supplemented by the divergence of a general superpotential plus all possible terms proportional to the equations of motion. We then derive systems of equations which we solve in order to prove several key results for spin-2 Fierz-Pauli theory, most notably that there are infinitely many conserved energy-momentum tensors derivable from the "improvement" method, and there are infinitely many conserved symmetric energy-momentum tensors that follow from specifying the Belinfante superpotential alone. This disproves several recent claims that the Belinfante tensor is uniquely associated to the Hilbert tensor in spin-2 Fierz-Pauli theory. We give two new energy-momentum tensors of this form. Most importantly, since there are infinitely many spin-2 energy-momentum tensors of this form, no meaningful or unique connection to Noether's first theorem can be claimed by application of the canonical Noether "improvement" method.

3.3.1 Motivation

The energy-momentum tensor is a fundamental object for a physical field theory. In electrodynamics, the Lorentz force law and Poynting's theorem are both expressed by the divergence of the uniquely accepted physical energy-momentum tensor $T^{\mu\nu}$. Several energy-momentum tensors that represent the linearized gravitational field exist in the literature. This causes an ambiguity in which the choice of energy-momentum tensor will impact the results of a given calculation and the conservation laws for the model as a whole. Selecting a unique $T^{\mu\nu}$ is problematic because Magnano and Sokolowski [140] showed that one cannot obtain a gauge invariant energy-momentum tensor for spin-2 Fierz-Pauli theory; an essential characteristic of the uniquely defined physical energy-momentum tensors for e.g. electrodynamics and Yang-Mills theory. As noted by Magnano and Sokolowski [140]:

> *"A gravitational energy–momentum tensor is highly desirable for a number of reasons. For instance, it is emphasized in* [6] *that such a genuinely local tensor is required for a detailed description of cosmological perturbations in the early universe. ... Furthermore, the metric stress tensor derived in* [6] *has a number of nice properties and according to the authors, their $T^{\mu\nu}$ is the correct energy–momentum tensor for the gravitational field. ... Applying a physically undeniable condition that the energy–momentum tensor should have the same gauge invariance as the field equations, we also conclude that this approach to gravity does not furnish a physically acceptable notion of gravitational energy density."*

The claim in [6] that the Hilbert (metric) energy-momentum tensor (3.79) is the correct physical expression seems to contradict the Hulse-Taylor 1993 Nobel prize in physics [107, 187], who used the equations Peters and Mathews developed [166] from the linearized Landau-Lifshitz energy-momentum pseudotensor (3.82) to model energy loss due to gravitational radiation of a binary pulsar system [131]; this observationally supported model uses a $T^{\mu\nu}$ which does not correspond to the Hilbert (metric) energy-momentum tensor. There exist many other energy-momentum pseudotensors for the gravitational field in general relativity (i.e. Weinberg [190], Papapetrou [164], Möller [152], Bergmann-Thomson [25], etc.) that can be linearized about the Minkowski background, further complicating the question as to which is physically significant in linearized gravity. Different energy-momentum tensors will be claimed to be the physical expression for spin-2 Fierz-Pauli theory in different publications, hence the need to address the non-uniqueness problem of energy-momentum tensors in linearized gravity. This was emphasized recently by Bičák and Schmidt [28], and to some degree our paper builds on their results.

Recent research has also proved that the Noether and Hilbert (metric) energy-momentum tensors are not, in general, equivalent [13]; that is, the physical energy-momentum tensor derived directly from Noether's first theorem [29] (symmetric, gauge invariant, conserved and trace-free) is not always equivalent to the Hilbert tensor in Minkowski spacetime for the same Lagrangian density. This further complicates the possibility of uniquely expressing an energy-momentum tensor for a physical theory. It is frequently asserted throughout the literature, however, that the canonical Noether energy-momentum tensor $T_C^{\mu\nu}$ (a non-symmetric, non-gauge invariant expression) derived using only the 4-parameter Poincaré translation, is the starting point for the derivation of various energy-momentum tensors found by the ad-hoc addition of the divergence of superpotentials, and terms proportional to the equations of motion; these ad-hoc additions are often referred to as "improvements". This is true in spin-2 Fierz-Pauli theory, where different expressions for $T^{\mu\nu}$ are claimed to have a connection to Noether's first theorem due to these ad-hoc "improvements" of the canonical Noether energy-momentum tensor. A good summary of some of these connections can be found in [184]. We will briefly review the most common expressions (Hilbert and Landau-Lifshitz energy-momentum tensors) to highlight this point. We start from the differential identity following from Noether's theorem [88, 124, 159],

$$\left(\frac{\partial\mathcal{L}}{\partial\Phi_A}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}+\partial_\mu\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}+\dots\right)\delta\Phi_A$$
$$+\partial_\mu\left(\eta^{\mu\nu}\mathcal{L}\delta x_\nu+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\delta\Phi_A+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\partial_\omega\delta\Phi_A-\left[\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\right]\delta\Phi_A+\dots\right)=0\,, \quad (3.75)$$

which is derived by asserting invariance of the action $S[\Phi_A(x_\alpha)]$ under infinitesimal changes in δx_ν and $\delta\Phi_A$. One can use the associated action symmetries of coordinates δx_ν and fields $\delta\Phi_A$ to derive on-shell conserved 'Noether currents' J^μ. The symmetries associated with the canonical Noether energy-momentum tensor are well known; they are the change in coordinates $\delta x_\nu = a_\nu$ (the 4-parameter translation of the 10 parameter Poincaré group), and transformation of fields $\delta\Phi_A = -(\partial^\nu\Phi_A)\delta x_\nu$. We begin with the spin-2 Fierz-Pauli Lagrangian density [74],

$$\mathcal{L}_{FP}=\frac{1}{4}[\partial_\alpha h^\beta_\beta\partial^\alpha h^\gamma_\gamma-\partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma}+2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha}-2\partial^\alpha h^\beta_\beta\partial^\gamma h_{\gamma\alpha}], \quad (3.76)$$

and the resulting spin-2 equation of motion $E^{\mu\nu}$ can be obtained from linearization of the Einstein tensor of general relativity. Equivalently, $E^{\mu\nu}$ follows from substitution of Equation (3.76) into the Euler-Lagrange expression in Equation (4.20),

$$E^{\mu\nu}=\frac{1}{2}[-\eta^{\mu\nu}\Box h+\Box h^{\mu\nu}+\partial^\mu\partial^\nu h-\partial_\lambda\partial^\nu h^{\mu\lambda}-\partial_\lambda\partial^\mu h^{\nu\lambda}+\eta^{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}]. \quad (3.77)$$

To derive the canonical Noether energy-momentum tensor for a second rank $h_{\mu\nu}$, from Equation (4.20) with transformation of coordinates $\delta x_\nu = a_\nu$ (the 4-parameter translation), and the transformation of fields $\delta\Phi_A = -(\partial^\nu\Phi_A)\delta x_\nu$, we have the canonical Noether energy-momentum tensor for a second rank $h_{\mu\nu}$, namely $T^{\mu\nu}_C = \eta^{\mu\nu}\mathcal{L}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}\partial^\nu h_{\alpha\beta}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega h_{\alpha\beta})}\partial_\omega\partial^\nu h_{\alpha\beta}+\left(\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega h_{\alpha\beta})}\right)\partial^\nu h_{\alpha\beta}+\dots$. For spin-2 Fierz-Pauli theory, we have a Lagrangian density in Equation (3.76) with terms of the form $\partial h\partial h$, thus the canonical Noether energy-momentum tensor reduces to $T^{\rho\sigma}_C = \eta^{\rho\sigma}\mathcal{L}_{FP}-\frac{\partial\mathcal{L}_{FP}}{\partial(\partial_\rho h_{\mu\nu})}\partial^\sigma h_{\mu\nu}$, thus we obtain using Equation (3.76) the canonical Noether energy-momentum tensor for spin-2 Fierz-Pauli theory,

$$T_C^{\rho\sigma} = \eta^{\rho\sigma}\mathcal{L}_{FP} - \frac{1}{2}[\partial^\rho h^\zeta_\zeta \partial^\sigma h^\mu_\mu - \partial^\rho h^{\mu\nu}\partial^\sigma h_{\mu\nu} - \partial^\gamma h^\zeta_\zeta \partial^\sigma h^\rho_\nu - \partial^\zeta h^\rho_\zeta \partial^\sigma h^\mu_\mu + 2\partial^\mu h^{\nu\rho}\partial^\sigma h_{\mu\nu}]. \quad (3.78)$$

On-shell conservation of the canonical Noether energy-momentum tensor is guaranteed by Noether's first theorem, which we can verify as $\partial_\rho T_C^{\rho\sigma} = E^{\lambda\gamma}\partial^\sigma h_{\lambda\gamma}$.

Using the Fierz-Pauli Lagrangian density in Equation (3.76) we will now derive the Hilbert energy-momentum tensor. The Hilbert energy-momentum tensor for a classical gauge theory in Minkowski space is defined in as $T_H^{\gamma\rho} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\gamma\rho}}\Big|_{g=\eta}$. The Hilbert energy-momentum tensor is derived from a Lagrangian density by replacing all ordinary derivatives with covariant derivatives $\partial \to \nabla$, replacing the Minkowski metric with the general metric tensor $\eta \to g$, and inserting the Jacobian term $\sqrt{-g}$. For spin-2 Fierz-Pauli theory we obtain the well known Hilbert energy-momentum tensor [168],

$$\begin{aligned} T_H^{\rho\sigma} = &\frac{1}{4}\eta^{\rho\sigma}[\partial_\alpha h^\beta_\beta \partial^\alpha h^\gamma_\gamma - \partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} + 2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha} + 2h^{\zeta\mu}\partial_\zeta\partial_\mu h^\beta_\beta] - \partial^\rho h_{\beta\alpha}\partial^\alpha h^{\beta\sigma} - \partial_\alpha h^\rho_\beta \partial^\sigma h^{\beta\alpha} + \partial_\alpha h^\rho_\beta \partial^\alpha h^{\sigma\beta} \\ &+ \partial_\zeta h^{\rho\lambda}\partial_\lambda h^{\sigma\zeta} - \partial_\zeta h^{\rho\sigma}\partial_\lambda h^{\lambda\zeta} - \frac{1}{2}\partial_\zeta h^{\rho\sigma}\partial^\zeta h^\lambda_\lambda - \frac{1}{2}\partial^\rho h^\beta_\beta \partial^\sigma h^\alpha_\alpha + \frac{1}{2}\partial^\rho h_{\beta\alpha}\partial^\sigma h^{\beta\alpha} + \frac{1}{2}\partial^\sigma h^\beta_\beta \partial^\alpha h^\rho_\alpha + \frac{1}{2}\partial^\rho h^\beta_\beta \partial^\alpha h^\sigma_\alpha \\ &+ h^{\rho\lambda}\partial_\zeta\partial_\lambda h^{\sigma\zeta} + h^{\sigma\lambda}\partial_\zeta\partial_\lambda h^{\rho\zeta} - h^{\rho\sigma}\partial_\zeta\partial_\lambda h^{\lambda\zeta} - h^{\lambda\zeta}\partial_\zeta\partial_\lambda h^{\rho\sigma} + \frac{1}{2}h^{\rho\sigma}\partial_\zeta\partial^\zeta h^\lambda_\lambda - \frac{1}{2}h^{\rho\mu}\partial^\sigma\partial_\mu h^\beta_\beta - \frac{1}{2}h^{\sigma\mu}\partial^\rho\partial_\mu h^\beta_\beta, \end{aligned} \quad (3.79)$$

which is conserved on-shell up to $\partial_\rho T_H^{\rho\sigma} = -2E^{\alpha\beta}\bar{\Gamma}^\sigma_{\alpha\beta}$, where $\bar{\Gamma}^\sigma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha h^\sigma_\beta + \partial_\beta h^\sigma_\alpha - \partial^\sigma h_{\beta\alpha})$ is the linearized Christoffel symbol of the second kind. Extracting (3.78) from (3.79) we can write the Hilbert tensor (3.79) as the canonical tensor (3.78) plus the divergence of a superpotential, and terms proportional to the equations of motion,

$$T_H^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\gamma \Psi_H^{[\rho\gamma]\sigma} - 2h^\sigma_\beta E^{\rho\beta}. \quad (3.80)$$

The Hilbert superpotential $\Psi_H^{[\rho\gamma]\sigma} = -\Psi_H^{[\gamma\rho]\sigma}$ found by rearranging (3.79) is,

$$\begin{aligned} \Psi_H^{[\rho\gamma]\sigma} = &\frac{1}{2}\eta^{\rho\sigma}h^{\gamma\alpha}\partial_\alpha h^\beta_\beta - \frac{1}{2}\eta^{\gamma\sigma}h^{\rho\alpha}\partial_\alpha h^\beta_\beta + \frac{1}{2}h^{\gamma\sigma}\partial^\rho h^\beta_\beta - \frac{1}{2}h^{\rho\sigma}\partial^\gamma h^\beta_\beta \\ &+ h^{\rho\lambda}\partial_\lambda h^{\gamma\sigma} - h^{\gamma\lambda}\partial_\lambda h^{\rho\sigma} + h^{\sigma\beta}\partial^\gamma h^\rho_\beta - h^{\sigma\beta}\partial^\rho h^\gamma_\beta. \end{aligned} \quad (3.81)$$

A superpotential $\Psi^{[\rho\gamma]\sigma}$ must be antisymmetric in two indices ($[\rho\gamma]$) because adding the divergence of a superpotential $\partial_\gamma\Psi^{[\rho\gamma]\sigma}$ to the canonical Noether expression must not affect the on-shell conservation ($\partial_\rho\partial_\gamma\Psi^{[\rho\gamma]\sigma} = 0$). This result was probably first noticed by Belinfante [23]. The Belinfante superpotential $b^{[\rho\gamma]\sigma}$ for spin-2 theory is exactly what is derived from the Hilbert superpotential $\Psi_H^{[\rho\gamma]\sigma}$, a result we will explore in detail; see for example (3.108). The relationship between the Belinfante superpotential, canonical Noether energy-momentum tensor and Hilbert energy-momentum tensor is well established [175, 95, 6, 32, 176, 76, 134, 133, 171]: the Hilbert energy-momentum tensor can be obtained by adding both the divergence of the Belinfante superpotential and specific terms proportional to the equations of motion to the canonical Noether energy-momentum tensor, which is consistent with the relationship in (3.80).

We will now introduce the linearized Landau-Lifshitz energy-momentum tensor [28],

$$\begin{aligned} T^{\mu\nu}_{LL} = \frac{3}{4}\eta^{\mu\nu}\partial_\alpha h\partial^\alpha h - \eta^{\mu\nu}\partial_\alpha h\partial_\beta h^{\alpha\beta} + \frac{1}{2}\eta^{\mu\nu}\partial^\lambda h^{\alpha\gamma}\partial_\gamma h_{\lambda\alpha} - \frac{1}{4}\eta^{\mu\nu}\partial^\alpha h^{\lambda\sigma}\partial_\alpha h_{\lambda\sigma} - \partial^\mu h\partial^\nu h \\ - \frac{3}{2}\partial_\alpha h^{\mu\nu}\partial^\alpha h + \partial_\alpha h^{\mu\nu}\partial_\beta h^{\alpha\beta} - \partial_\alpha h^{\mu\alpha}\partial_\beta h^{\nu\beta} + \frac{1}{2}\partial^\mu h^{\lambda\sigma}\partial^\nu h_{\lambda\sigma} + \partial_\lambda h^{\mu\alpha}\partial^\lambda h^\nu{}_\alpha \\ + (\partial_\alpha h^{\mu\alpha}\partial^\nu h + \partial_\alpha h^{\nu\alpha}\partial^\mu h) + \frac{1}{2}(\partial^\mu h^{\nu\gamma}\partial_\gamma h + \partial^\nu h^{\mu\gamma}\partial_\gamma h) - (\partial^\mu h_{\beta\gamma}\partial^\gamma h^{\nu\beta} + \partial^\nu h_{\beta\gamma}\partial^\gamma h^{\mu\beta}). \end{aligned} \tag{3.82}$$

This energy-momentum tensor can also be expressed in terms of (3.78) and terms proportional to the equations of motion (3.77) as [184],

$$T^{\mu\nu}_{LL} = T^{\mu\nu}_C + \partial_\alpha \Psi^{[\mu\alpha]\nu}_{LL} + hE^{\mu\nu} - 2h^\nu_\beta E^{\mu\beta}, \tag{3.83}$$

where the Landau-Lifshitz superpotential is,

$$\begin{aligned} \Psi^{[\mu\alpha]\nu}_{LL} = \frac{1}{2}[\eta^{\mu\nu}h\partial^\alpha h - \eta^{\nu\alpha}h\partial^\mu h + \eta^{\nu\alpha}h\partial_\beta h^{\mu\beta} - \eta^{\mu\nu}h\partial_\beta h^{\alpha\beta} + h\partial^\mu h^{\nu\alpha} - h\partial^\alpha h^{\mu\nu}] \\ + h^{\nu\alpha}\partial^\mu h - h^{\mu\nu}\partial^\alpha h + h^{\mu\nu}\partial_\beta h^{\alpha\beta} - h^{\nu\alpha}\partial_\lambda h^{\mu\lambda} + h^\nu_\beta\partial^\alpha h^{\mu\beta} - h^\nu_\beta\partial^\mu h^{\beta\alpha}. \end{aligned} \tag{3.84}$$

Both the Hilbert $T^{\rho\sigma}_H$ and Landau-Lifshitz $T^{\mu\nu}_{LL}$ energy-momentum tensors can be obtained by starting from the canonical Noether energy-momentum tensor of spin-2 Fierz-Pauli theory $T^{\rho\sigma}_C$, then ad-hoc adding the divergence of a superpotential, and terms proportional to the equations of motion (3.77). This is frequently used to assert that these results can in some way be derived from Noether's first theorem. For example, in the Padmanabhan-Deser debate, [163, 41, 63, 40, 16], Padmanabhan asserted [163] that for self coupling of the spin-2 energy-momentum tensor, one can add infinitely many different superpotentials to the canonical Noether tensor, thus Noether's procedure cannot be used to determine the energy-momentum tensor. Subsequent authors [63, 16], asserted that Noether's theorem can be used by adding the Belinfante superpotential and additional terms proportional to the equations of motion to the canonical Noether tensor to obtain $T_H^{\rho\sigma}$ in (3.79). But this ad-hoc addition of terms can also be used to obtain other published expressions, such as the Landau-Lifshitz in $T_{LL}^{\mu\nu}$ (3.82). Since known energy-momentum tensors can be obtained by adding the ad-hoc correction terms to the canonical Noether tensor, this "improvement" process is portrayed as a meaningful connection of any such energy-momentum tensor to Noether's first theorem. However, some honest discussion of these ad-hoc "improvements" can be found in the literature, such as statements made by Forger and Römer [76]:

> *"There is a long history of attempts to cure these diseases and arrive at the physically correct energy-momentum tensor $T^{\mu\nu}$ by adding judiciously chosen "improvement" terms to [$T_C^{\mu\nu}$]".* They go on to say: *"However, all these methods of defining improved energy-momentum tensors are largely "ad hoc" prescriptions focussed on special models of field theory, often geared to the needs of quantum field theory and ungeometric in spirit".*

We point out that Bessel-Hagen (a contemporary and colleague of Noether) first showed how to derive the physical energy-momentum tensor directly from Noether's first theorem without the need for any such ad-hoc "improvements" in 1921 [26]. This result was determined independently by later authors ([70, 39, 153], to name a few) and summarized in [29]. Furthermore, it has recently been shown that the Noether and Hilbert energy-momentum tensor are not, in general, equivalent [13], which further emphasizes the need for the investigation into the relationship between tensors which are derived directly from Noether's first theorem, and those which can only be obtained after adding the divergence of a superpotential and terms proportional to the equations of motion for a particular theory. This will be a subject that we address in this article.

Bičák and Schmidt explored the non-uniqueness of the energy-momentum tensors in linearized gravity in a recent article [28] (Bičák has long be interested in this question [27]). They used the Fock method for deriving an energy-momentum tensor [75, 109], which considers general expressions of terms with free coefficients for $T^{\rho\sigma}$. In particular, Bičák and Schmidt consider all possible terms of the form $\partial h \partial h$; we will denote their Fock energy-momentum tensor as $T_{BS}^{\rho\sigma}$,

$$
\begin{aligned}
T_{BS}^{\rho\sigma} &= b_1\partial_\alpha h^{\rho\sigma}\partial_\beta h^{\alpha\beta} + b_2\partial_\alpha h^{\rho\sigma}\partial^\alpha h + b_3\partial_\alpha h^{\rho\alpha}\partial_\beta h^{\sigma\beta} + b_4\partial_\alpha h^{\rho}{}_{\beta}\partial^\alpha h^{\sigma\beta} + b_5\partial_\alpha h^{\rho\beta}\partial_\beta h^{\sigma\alpha} \\
&+ b_6\partial^\rho h\partial^\sigma h + b_7\partial^\rho h_{\alpha\beta}\partial^\sigma h^{\alpha\beta} + b_{8_i}\partial^\rho h^{\sigma\alpha}\partial_\alpha h + b_{8_{ii}}\partial^\sigma h^{\rho\alpha}\partial_\alpha h + b_{9_i}\partial^\rho h\partial_\alpha h^{\sigma\alpha} + b_{9_{ii}}\partial^\sigma h\partial_\alpha h^{\rho\alpha} \\
&+ b_{10_i}\partial^\rho h^{\sigma\alpha}\partial^\beta h_{\alpha\beta} + b_{10_{ii}}\partial^\sigma h^{\rho\alpha}\partial^\beta h_{\alpha\beta} + b_{11_i}\partial^\rho h_{\alpha\beta}\partial^\alpha h^{\sigma\beta} + b_{11_{ii}}\partial^\sigma h_{\alpha\beta}\partial^\alpha h^{\rho\beta} \\
&+ c_1\eta^{\rho\sigma}\partial_\alpha h\partial^\alpha h + c_2\eta^{\rho\sigma}\partial_\alpha h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + c_3\eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\lambda h^{\lambda}{}_{\beta} + c_4\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} + c_5\eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\beta h.
\end{aligned} \tag{3.85}
$$

This Fock energy-momentum tensor appears in Equation 8 of their article [28]. In (3.85) we separate terms proportional to the Minkowski metric $\eta^{\rho\sigma}$ with coefficients c_n. The authors use this to prove some very interesting results which we have also verified, such as the uniqueness of the Landau-Liftshitz tensor as the conserved and symmetric expression that follows from (3.85). The problem is that (3.85) is not the most general expression, because many conserved energy-momentum tensors have terms of the form $h\partial\partial h$, such as the Hilbert tensor (3.79). The appearance of such terms ($h\partial\partial h$) greatly complicates the resulting linear system of coefficients. To accommodate these additional terms we will take a similar approach to [28], but instead we consider the most general possible Fock energy-momentum tensor for spin-2 Fierz-Pauli theory. This will be used to complete several proofs regarding the energy-momentum tensors in linearized gravity. In particular, we will consider the most general system which can be obtained by adding the divergence of a superpotential and terms proportion to the equations of motion to the canonical Noether energy-momentum tensor $T_C^{\mu\nu}$. Using this expression we will prove that there are infinitely many conserved tensors that can be obtained by the ad-hoc addition of these terms to $T_C^{\mu\nu}$, and that there are infinitely many symmetric conserved energy-momentum tensors following from the Belinfante improvement procedure alone. We argue that these results show that no meaningful connection to Noether's first theorem exists from the superpotential approach; if a tensor is not directly derived from Noether's first theorem, then it simply is not derived from Noether's first theorem, and no amount of ad-hoc "improvements" can change this fact.

3.3.2 The most general Fock energy-momentum tensor for linearized gravity

Since Bičák and Schmidt in [28] consider (3.85), which is not the most general energy-momentum tensor for linearized gravity, as it does not include e.g. the Hilbert energy-momentum tensor in (3.79). We will begin our proofs with the most general expression that also includes terms of the form $h\partial\partial h$,

$$
\begin{aligned}
T^{\rho\sigma} = {} & b_1 \partial_\alpha h^{\rho\sigma} \partial_\beta h^{\alpha\beta} + b_2 \partial_\alpha h^{\rho\sigma} \partial^\alpha h + b_3 \partial_\alpha h^{\rho\alpha} \partial_\beta h^{\sigma\beta} + b_4 \partial_\alpha h^{\rho}{}_{\beta} \partial^\alpha h^{\sigma\beta} + b_5 \partial_\alpha h^{\rho\beta} \partial_\beta h^{\sigma\alpha} \\
& + b_6 \partial^\rho h \partial^\sigma h + b_7 \partial^\rho h_{\alpha\beta} \partial^\sigma h^{\alpha\beta} + b_{8_i} \partial^\rho h^{\sigma\alpha} \partial_\alpha h + b_{8_{ii}} \partial^\sigma h^{\rho\alpha} \partial_\alpha h + b_{9_i} \partial^\rho h \partial_\alpha h^{\sigma\alpha} + b_{9_{ii}} \partial^\sigma h \partial_\alpha h^{\rho\alpha} \\
& + b_{10_i} \partial^\rho h^{\sigma\alpha} \partial^\beta h_{\alpha\beta} + b_{10_{ii}} \partial^\sigma h^{\rho\alpha} \partial^\beta h_{\alpha\beta} + b_{11_i} \partial^\rho h_{\alpha\beta} \partial^\alpha h^{\sigma\beta} + b_{11_{ii}} \partial^\sigma h_{\alpha\beta} \partial^\alpha h^{\rho\beta} \\
& + c_1 \eta^{\rho\sigma} \partial_\alpha h \partial^\alpha h + c_2 \eta^{\rho\sigma} \partial_\alpha h_{\beta\lambda} \partial^\alpha h^{\beta\lambda} + c_3 \eta^{\rho\sigma} \partial_\alpha h^{\alpha\beta} \partial_\lambda h^{\lambda}{}_{\beta} + c_4 \eta^{\rho\sigma} \partial_\alpha h_{\lambda\beta} \partial^\lambda h^{\alpha\beta} + c_5 \eta^{\rho\sigma} \partial_\alpha h^{\alpha\beta} \partial_\beta h \\
& + d_1 h^{\rho\sigma} \partial_\alpha \partial^\alpha h + d_2 h^{\rho\sigma} \partial_\alpha \partial_\beta h^{\alpha\beta} + d_3 h \partial_\alpha \partial^\alpha h^{\rho\sigma} + d_4 h^{\alpha\beta} \partial_\alpha \partial_\beta h^{\rho\sigma} + d_{5_i} h^{\rho\alpha} \partial^\beta \partial_\beta h^{\sigma}{}_{\alpha} + d_{5_{ii}} h^{\sigma\alpha} \partial^\beta \partial_\beta h^{\rho}{}_{\alpha} \\
& + d_{6_i} h^{\rho\alpha} \partial_\alpha \partial_\beta h^{\sigma\beta} + d_{6_{ii}} h^{\sigma\alpha} \partial_\alpha \partial_\beta h^{\rho\beta} + d_7 h \partial^\rho \partial^\sigma h + d_8 h_{\alpha\beta} \partial^\rho \partial^\sigma h^{\alpha\beta} \\
& + d_{9_i} h^{\rho\alpha} \partial^\sigma \partial_\alpha h + d_{9_{ii}} h^{\sigma\alpha} \partial^\rho \partial_\alpha h + d_{10_i} h^{\rho\alpha} \partial^\sigma \partial^\beta h_{\alpha\beta} + d_{10_{ii}} h^{\sigma\alpha} \partial^\rho \partial^\beta h_{\alpha\beta} \\
& + d_{11_i} h \partial^\rho \partial_\alpha h^{\sigma\alpha} + d_{11_{ii}} h \partial^\sigma \partial_\alpha h^{\rho\alpha} + d_{12_i} h_{\alpha\beta} \partial^\rho \partial^\alpha h^{\sigma\beta} + d_{12_{ii}} h_{\alpha\beta} \partial^\sigma \partial^\alpha h^{\rho\beta} \\
& + a_1 \eta^{\rho\sigma} h_{\alpha\beta} \partial^\alpha \partial^\beta h + a_2 \eta^{\rho\sigma} h \partial_\alpha \partial_\beta h^{\alpha\beta} + a_3 \eta^{\rho\sigma} h_{\alpha\beta} \partial^\alpha \partial_\lambda h^{\lambda\beta} + a_4 \eta^{\rho\sigma} h \partial_\alpha \partial^\alpha h + a_5 \eta^{\rho\sigma} h_{\alpha\beta} \partial_\lambda \partial^\lambda h^{\alpha\beta},
\end{aligned}
\tag{3.86}
$$

where we separate terms $h\partial\partial h$ that are proportional to the Minkowski metric $\eta^{\rho\sigma}$ with coefficients a_n. The general idea of the Fock method is to take the divergence $\partial_\rho T^{\rho\sigma}$ and solve for the free coefficients in front of each term such that the resulting energy-momentum tensor is conserved on-shell. These coefficients can be solved to impose various other properties, such as symmetry or tracelessness. Terms which must have an identical coefficient for a symmetric expression with subscripts (i) and (ii). For example, terms b_{8_i} and $b_{8_{ii}}$ form a symmetric pair when $b_{8_i} = b_{8_{ii}}$. Terms b_n correspond to terms $\partial h \partial h$ that are not proportional to Minkowski $\eta^{\rho\sigma}$, and terms d_n correspond to terms $h\partial\partial h$ that are not proportional to Minkowski $\eta^{\rho\sigma}$. The four sets of free coefficients make the proofs and linear systems of equations easier to follow.

The general idea of the Fock method, to take the divergence $\partial_\rho T^{\rho\sigma}$ of (3.86) and solve for coefficients that allow for a conserved expression up to $E^{\mu\nu}$ in (3.77). This will also include terms proportional to the trace of the equation of motion (3.77) which we obtain from $\mathbf{E} = \eta_{\mu\nu} E^{\mu\nu}$,

$$
\mathbf{E} = \partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h. \tag{3.87}
$$

We wish to explore the most general expression (3.86), and its relationship to the canonical Noether energy-momentum tensor (3.78), supplemented by the most general possible superpotential and terms proportional to the equations of motion. There are six possible terms proportional to the equations of motion $hE^{\rho\sigma}$, $h^{\rho}_{\alpha} E^{\sigma\alpha}$, $h^{\rho\sigma}\mathbf{E}$, $h^{\sigma}_{\alpha} E^{\rho\alpha}$, $\eta^{\rho\sigma} h\mathbf{E}$ and $\eta^{\rho\sigma} h_{\alpha\beta} E^{\alpha\beta}$, each of which will be general up to a coefficient ζ_n. Therefore what we will solve for is the most general possible case where one supplements the canonical Noether energy-momentum tensor of spin-2 Fierz-Pauli theory (3.78) by the divergence of a superpotential and terms proportional to the equations of motion (i.e. we will obtain solutions of the form $T^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha \Psi^{[\rho\alpha]\sigma} + \zeta_1 h E^{\rho\sigma} + \zeta_2 h^\rho_\alpha E^{\sigma\alpha} + \zeta_3 h^{\rho\sigma}\mathbf{E} + \zeta_4 h^\sigma_\alpha E^{\rho\alpha} + \zeta_5 \eta^{\rho\sigma} h\mathbf{E} + \zeta_6 \eta^{\rho\sigma} h_{\alpha\beta} E^{\alpha\beta}$). To do this, we must re-express (3.86) in terms of the 28 possible terms from the ζ_n expressions, and all possible superpotential terms. This process is non-trivial, so we will now explain how one must re-express these terms.

In total there are 43 terms in (3.86). Using the identity $A\partial B = \partial(AB) - B\partial A$ based on the product rule for each of the quadratic terms b_n and c_n, we can express all terms of the form $\partial h \partial h$ as $h\partial\partial h$ (terms presented in the equations of motion) plus terms under a total divergence of the form $\partial[h\partial h]$ (which contribute to the superpotential). For example we can re-write the b_1 term in (3.86) as $b_1 \partial_\alpha h^{\rho\sigma} \partial_\beta h^{\alpha\beta} = b_1 \partial_\alpha [h^{\rho\sigma} \partial_\beta h^{\alpha\beta}] - b_1 h^{\rho\sigma} \partial_\alpha \partial_\beta h^{\alpha\beta}$. No terms can be neglected as in the case of boundary terms in the action when deriving the equations of motion; total divergences in this derivation contribute to the superpotential. We give the result of this process in (3.88), below. Terms of the form $h\partial\partial h$ will factor into combinations of the equations of motion in (3.77) and (3.87). The total divergence term will result in the divergence of the superpotential term of the form $\partial_\alpha \Psi^{[\rho\alpha]\sigma}$. As discussed earlier, the superpotential $\Psi^{[\rho\alpha]\sigma}$ must be antisymmetric in $[\rho\alpha]$ so that the total expression for $T^{\rho\sigma}$ is conserved on-shell via $\partial_\rho \partial_\alpha \Psi^{[\rho\alpha]\sigma} = 0$ (since $T_C^{\rho\sigma}$ is independently conserved on-shell due to Noether's first theorem, and all of the terms proportional to the equations of motion trivially do not impact on-shell conservation).

The complicated part is that some b_n and c_n terms can be combined using the identity $A\partial B = \partial(AB) - B\partial A$ in two different ways. In addition there are terms a_n and d_n that the identity $A\partial B = \partial(AB) - B\partial A$ can be applied to twice, contributing two pieces to the superpotential and a different piece to the equations of motion. All of these possibilities must be accounted for in the most general linear system: one cannot simply split these possibilities by a numerical coefficient such as $\frac{1}{2}$, because the relative contribution can be uneven, such as in the case of the b_2 term in $T_{LL}^{\mu\nu}$ (3.82), $-\frac{3}{2}\partial_\alpha h^{\mu\nu}\partial^\alpha h$. In general, we will split such terms in the form $b_n = B_n + \bar{B}_n$. For example term $b_2 = B_2 + \bar{B}_2$ must be split. This is because it can make contributions $B_2 \partial_\alpha h^{\rho\sigma} \partial^\alpha h = B_2 \partial_\alpha [h^{\rho\sigma} \partial^\alpha h] - B_2 h^{\rho\sigma} \partial_\alpha \partial^\alpha h$ and contribution $\bar{B}_2 \partial_\alpha h^{\rho\sigma} \partial^\alpha h = \bar{B}_2 \partial_\alpha [h \partial^\alpha h^{\rho\sigma}] - \bar{B}_2 h \partial_\alpha \partial^\alpha h^{\rho\sigma}$. For this reason terms b_2 can contribute multiple different terms to the superpotential of a particular energy-momentum tensor, as seen in the general result (3.88), and as emphasized by the Landau-Lifshitz example. The exact splitting of each coefficient can be nontrivial and must be solved for as part of the general system of linear equations (which we summarize in Appendix A).

However, the majority of the terms in (3.86) cannot be split, because they will either not contribute to one of the 6 possible equations of motion (e.g. terms such as $h_{\alpha\beta}\partial^{\alpha}\partial^{\beta}h^{\rho\sigma}$), or they will produce a term symmetric in $(\rho\alpha)$ in the total divergence which cannot be incorporated into the superpotential $\Psi^{[\rho\alpha]\sigma}$ (e.g. terms with a ∂^{ρ} total derivative). The third possibility is both applications of the identity $A\partial B = \partial(AB) - B\partial A$ yield the same result, thus they recombine and no splitting is necessary. Taking this all into account, there are 9 terms which must be split due to multiple possible contributions in the most general system. They are a_1, a_2, b_2, b_4, c_5, d_1, d_3, d_{5_i} and $d_{5_{ii}}$. Each of these 9 coefficients is split in the form $a_n = A_n + \bar{A}_n$, $b_n = B_n + \bar{B}_n$, $c_n = C_n + \bar{C}_n$ and $d_n = D_n + \bar{D}_n$. Using these conditions on (3.86) and the identity $A\partial B = \partial(AB) - B\partial A$ accordingly we are left with the general energy-momentum tensor:

$$
\begin{aligned}
T^{\rho\sigma} = {} & (b_7 - d_8)\partial^\sigma h_{\alpha\beta}\partial^\rho h^{\alpha\beta} + c_4\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} + (b_{10_i} - d_{12_i})\partial_\alpha[\eta^{\rho\alpha}h^{\sigma\beta}\partial^\omega h_{\omega\beta}] \\
& + \partial_\alpha[(B_2 + \bar{D}_1 - \bar{D}_3)h^{\rho\sigma}\partial^\alpha h + b_{9_i}h^{\sigma\alpha}\partial^\rho h + (\bar{B}_2 - \bar{D}_1 + \bar{D}_3)h\partial^\alpha h^{\rho\sigma} + b_{8_i}h\partial^\rho h^{\sigma\alpha} \\
& + (b_1 - d_4)h^{\rho\sigma}\partial_\beta h^{\alpha\beta} + b_3 h^{\sigma\alpha}\partial_\beta h^{\rho\beta} + d_4 h^{\alpha\beta}\partial_\beta h^{\rho\sigma} + b_5 h^{\rho\beta}\partial_\beta h^{\sigma\alpha} \\
& + d_{12_i}h^\alpha{}_\beta\partial^\rho h^{\sigma\beta} + (B_4 + \bar{D}_{5_i} - \bar{D}_{5_{ii}})h^\rho{}_\beta\partial^\alpha h^{\sigma\beta} + d_{12_{ii}}h^\alpha{}_\beta\partial^\sigma h^{\rho\beta} + b_{11_{ii}}h^\rho{}_\beta\partial^\sigma h^{\alpha\beta} \\
& + (\bar{B}_4 - \bar{D}_{5_i} + \bar{D}_{5_{ii}})h^\sigma{}_\beta\partial^\alpha h^{\rho\beta} + b_{11_i}h^\sigma{}_\beta\partial^\rho h^{\alpha\beta} + b_6\eta^{\sigma\alpha}h\partial^\rho h + c_1\eta^{\rho\sigma}h\partial^\alpha h \\
& + d_8\eta^{\sigma\alpha}h^{\omega\beta}\partial^\rho h_{\omega\beta} + c_2\eta^{\rho\sigma}h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + b_{8_{ii}}\eta^{\sigma\alpha}h^{\rho\omega}\partial_\omega h + (C_5 + \bar{A}_1 - \bar{A}_2)\eta^{\rho\sigma}h^{\alpha\beta}\partial_\beta h \\
& + (\bar{C}_5 - \bar{A}_1 + \bar{A}_2)\eta^{\rho\sigma}h\partial_\omega h^{\omega\alpha} + b_{9_{ii}}\eta^{\sigma\alpha}h\partial_\omega h^{\rho\omega} + (b_{10_{ii}} - d_{12_{ii}})\eta^{\sigma\alpha}h^{\rho\omega}\partial^\beta h_{\omega\beta} + c_3\eta^{\rho\sigma}h^{\alpha\beta}\partial_\lambda h^\lambda{}_\beta] \\
& + h[(d_7 - b_6)\partial^\rho\partial^\sigma h + (D_3 - \bar{B}_2 + \bar{D}_1)\partial_\alpha\partial^\alpha h^{\rho\sigma} + (d_{11_i} - b_{8_i})\partial^\rho\partial_\alpha h^{\sigma\alpha} \\
& + (d_{11_{ii}} - b_{9_{ii}})\partial^\sigma\partial_\alpha h^{\rho\alpha} + \bar{M}_3\eta^{\rho\sigma}\partial_\alpha\partial_\beta h^{\alpha\beta} + \bar{M}_4\eta^{\rho\sigma}\partial_\alpha\partial^\alpha h] \\
& + h^\rho_\alpha[(d_{10_i} + d_{12_{ii}} - b_{10_{ii}} - b_{11_{ii}})\partial^\sigma\partial_\beta h^{\alpha\beta} + (d_{9_i} - b_{8_{ii}})\partial^\sigma\partial^\alpha h + (d_{6_i} - b_5)\partial^\alpha\partial_\beta h^{\sigma\beta} \\
& + (D_{5_i} + \bar{D}_{5_{ii}} - B_4)\partial^\beta\partial_\beta h^{\sigma\alpha} + \bar{M}_1\eta^{\sigma\alpha}\partial_\omega\partial_\beta h^{\omega\beta} + \bar{M}_2\eta^{\sigma\alpha}\partial_\omega\partial^\omega h] \\
& + h^{\rho\sigma}[M_1\partial_\alpha\partial_\beta h^{\alpha\beta} + M_2\partial_\alpha\partial^\alpha h] \\
& + \eta^{\rho\sigma}h[M_3\partial_\alpha\partial_\beta h^{\alpha\beta} + M_4\partial_\alpha\partial^\alpha h] \\
& + h^\sigma_\alpha[(d_{10_{ii}} + d_{12_i} - b_{10_i} - b_{11_i})\partial^\rho\partial_\beta h^{\alpha\beta} + (d_{9_{ii}} - b_{9_i})\partial^\rho\partial^\alpha h + (d_{6_{ii}} - b_3)\partial^\alpha\partial_\beta h^{\rho\beta} \\
& + (D_{5_{ii}} + \bar{D}_{5_i} - \bar{B}_4)\partial^\beta\partial_\beta h^{\rho\alpha} + \hat{M}_1\eta^{\rho\alpha}\partial_\omega\partial_\beta h^{\omega\beta} + \hat{M}_2\eta^{\rho\alpha}\partial_\omega\partial^\omega h] \\
& + \eta^{\rho\sigma}h_{\alpha\beta}[(a_5 - c_2)\partial_\lambda\partial^\lambda h^{\alpha\beta} + (A_1 - C_5 + \bar{A}_2)\partial^\alpha\partial^\beta h + \frac{1}{2}(a_3 - c_3)\partial_\omega\partial^\alpha h^{\omega\beta} \\
& + \frac{1}{2}(a_3 - c_3)\partial_\omega\partial^\beta h^{\omega\alpha} + \hat{M}_3\eta^{\alpha\beta}\partial_\omega\partial_\gamma h^{\omega\gamma} + \hat{M}_4\eta^{\alpha\beta}\partial_\omega\partial^\omega h]. \qquad (3.88)
\end{aligned}
$$

The M_n coefficients are required because after separating terms, an additional splitting is required for terms proportional to the Minkowski metric $\eta^{\rho\sigma}$ that can each be separated in 3 possible ways across the equations of motion. They were separated according to:

$$d_2 + d_4 - b_1 = M_1 + \bar{M}_1 + \hat{M}_1 \tag{3.89}$$

$$D_1 - B_2 + \bar{D}_3 = M_2 + \bar{M}_2 + \hat{M}_2 \tag{3.90}$$

$$A_2 - \bar{C}_5 + \bar{A}_1 = M_3 + \bar{M}_3 + \hat{M}_3 \tag{3.91}$$

$$a_4 - c_1 = M_4 + \bar{M}_4 + \hat{M}_4 \tag{3.92}$$

Solving the linear system of equations for the coefficients in (3.88) and imposing the conditions on $T^{\rho\sigma}$ gives insight into the most general energy-momentum tensor for spin-2 written in terms of the canonical Noether $T_C^{\rho\sigma}$ plus the divergence of a superpotential and terms proportional to the equations of motion. The bottom 6 groups of terms in (3.88) are the six possible terms proportional to the equations of motion. The total divergence on the second line of (3.88) is sorted according to pairs which will form the most general possible superpotential $\Psi^{[\rho\alpha]\sigma}$ for linearized gravity according to the most general Fock expression in (3.86).

We note the 3 terms, separated at the top of the (3.88) expression, cannot be fit into either the most general superpotential or terms proportional to the equations of motion. The first two terms $(b_7 - d_8)\partial^\sigma h_{\alpha\beta}\partial^\rho h^{\alpha\beta}$ and $c_4\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta}$ are found in the canonical energy-momentum tensor (3.78) which, in part, explains why ad-hoc addition of the divergence of a superpotential and terms proportional to the equations of motion can seemingly be used to obtain any published energy-momentum tensor. The final term $(b_{10_i} - d_{12_i})\partial_\alpha[\eta^{\rho\alpha}h^{\sigma\beta}\partial^\omega h_{\omega\beta}]$ is symmetric in $(\rho\alpha)$ in the total divergence thus cannot be combined to the superpotential, and it cannot be combined into any of the equations of motion. This will produce an independent equation in our general linear system.

In addition the manifestly symmetric form follows from the symmetry conditions:

$$b_n = b_{n_i} = b_{n_{ii}} \tag{3.93}$$

$$d_n = d_{n_i} = d_{n_{ii}} \tag{3.94}$$

We will return to these symmetry conditions later in the article.

3.3.3 The most general canonical Noether energy-momentum tensor supplemented by ad-hoc "improvements"

We will now ask the general question, namely, what is the most general possible superpotential, and terms proportional to the equations of motion, that can be added ad-hoc to the canonical

Noether energy-momentum tensor (3.78) in order to obtain a general system of on-shell conserved energy-momentum tensors in linearized gravity. In order to not impact the flow of the text, we present the system of linear equations of the coefficients resulting from this process in Appendix A. Any spin-2 energy-momentum tensor that can be obtained by "improving" the canonical Noether tensor (3.78) can be determined by solving this system of linear equations; the exact superpotential and terms proportional to the equations of motion (3.77) trivially follow.

The aforementioned first 3 terms in (3.88) are independent conditions that must be solved. The first two, as appearing in (3.78), must be related to the coefficients of the spin-2 Fierz-Pauli canonical energy-momentum tensor (which appears the same in both (3.79) and (3.82) as well). The third term must be independently satisfied. Therefore to obtain the canonical $T_C^{\rho\sigma}$ (3.78) in (3.88) we minimally require the conditions in equations (3.136) to (3.138) in Appendix A.

The remaining terms in (3.78) must be extracted from the general system of coefficients. Since (3.78) has coefficients from (3.86) that are $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$, $c_4 = \frac{1}{2}$, $c_5 = -\frac{1}{2}$, $b_6 = -\frac{1}{2}$, $b_7 = \frac{1}{2}$, $b_{8_{ii}} = \frac{1}{2}$, $b_{9_{ii}} = \frac{1}{2}$, and $b_{11_{ii}} = -1$, and we have already solved for c_4 and b_7, we only need to extract the remaining coefficients. Thus we need to shift the coefficients in (3.88) by $c_1 \to c_1 - \frac{1}{4}$, $c_2 \to c_2 + \frac{1}{4}$, $c_5 \to c_5 + \frac{1}{2}$, $b_6 \to b_6 + \frac{1}{2}$, $b_{8_{ii}} \to b_{8_{ii}} - \frac{1}{2}$, $b_{9_{ii}} \to b_{9_{ii}} - \frac{1}{2}$, and $b_{11_{ii}} \to b_{11_{ii}} + 1$ to exactly obtain (3.78) in (3.88). This modifies the c_5 splitting condition, thus we now have the splitting conditions from (3.139) to (3.147).

These coefficient shifts (obtained by extracting the canonical Noether energy-momentum tensor) modify the general superpotential in (3.88) to,

$$\begin{aligned}\Psi^{[\rho\alpha]\sigma} = {} & (B_2 + \bar{D}_1 - \bar{D}_3)h^{\rho\sigma}\partial^\alpha h + b_{9_i}h^{\sigma\alpha}\partial^\rho h + (\bar{B}_2 - \bar{D}_1 + \bar{D}_3)h\partial^\alpha h^{\rho\sigma} + b_{8_i}h\partial^\rho h^{\sigma\alpha} \\ & + (b_1 - d_4)h^{\rho\sigma}\partial_\beta h^{\alpha\beta} + b_3 h^{\sigma\alpha}\partial_\beta h^{\rho\beta} + d_4 h^{\alpha\beta}\partial_\beta h^{\rho\sigma} + b_5 h^{\rho\beta}\partial_\beta h^{\sigma\alpha} \\ & + d_{12_i}h^\alpha{}_\beta\partial^\rho h^{\sigma\beta} + (B_4 + \bar{D}_{5_i} - \bar{D}_{5_{ii}})h^\rho{}_\beta\partial^\alpha h^{\sigma\beta} + d_{12_{ii}}h^\alpha{}_\beta\partial^\sigma h^{\rho\beta} + (b_{11_{ii}} + 1)h^\rho{}_\beta\partial^\sigma h^{\alpha\beta} \\ & + (\bar{B}_4 - \bar{D}_{5_i} + \bar{D}_{5_{ii}})h^\sigma{}_\beta\partial^\alpha h^{\rho\beta} + b_{11_i}h^\sigma{}_\beta\partial^\rho h^{\alpha\beta} + (b_6 + \frac{1}{2})\eta^{\sigma\alpha}h\partial^\rho h + (c_1 - \frac{1}{4})\eta^{\rho\sigma}h\partial^\alpha h \\ & + d_8\eta^{\sigma\alpha}h^{\omega\beta}\partial^\rho h_{\omega\beta} + (c_2 + \frac{1}{4})\eta^{\rho\sigma}h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + (b_{8_{ii}} - \frac{1}{2})\eta^{\sigma\alpha}h^{\rho\omega}\partial_\omega h + (C_5 + \bar{A}_1 - \bar{A}_2)\eta^{\rho\sigma}h^{\alpha\beta}\partial_\beta h \\ & + (\bar{C}_5 - \bar{A}_1 + \bar{A}_2)\eta^{\rho\sigma}h\partial_\omega h^{\omega\alpha} + (b_{9_{ii}} - \frac{1}{2})\eta^{\sigma\alpha}h\partial_\omega h^{\rho\omega} + (b_{10_{ii}} - d_{12_{ii}})\eta^{\sigma\alpha}h^{\rho\omega}\partial^\beta h_{\omega\beta} + c_3\eta^{\rho\sigma}h^{\alpha\beta}\partial_\lambda h^\lambda{}_\beta.\end{aligned} \tag{3.95}$$

We note that antisymmetric pairs are sorted throughout this expression. Using the superpotential condition $\partial_\rho\partial_\alpha\Psi^{[\rho\alpha]\sigma} = 0$ we therefore straightforwardly obtain the conditions for the antisymmetric superpotential in equations (3.148) to (3.159).

Imposing conditions (3.136) to (3.138), (3.139) to (3.147) and (3.148) to (3.159) from Appendix A on (3.88) and writing the equation of motion coefficients in terms of ζ_n we obtain the desired compact result,

$$T^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha \Psi^{[\rho\alpha]\sigma} + \zeta_1 h E^{\rho\sigma} + \zeta_2 h^\rho_\alpha E^{\sigma\alpha} + \zeta_3 h^{\rho\sigma}\mathbf{E} + \zeta_4 h^\sigma_\alpha E^{\rho\alpha} + \zeta_5 \eta^{\rho\sigma} h\mathbf{E} + \zeta_6 \eta^{\rho\sigma} h_{\alpha\beta} E^{\alpha\beta}, \quad (3.96)$$

where the equations of motion have slightly modified coefficients when compared to those in (3.88) due to the coefficient shifts above. The resulting system of linear equations for the equations of motion ζ_n are given in Appendix A as (3.160) to (3.187). The M coefficients in (3.89) to (3.92) have also been modified from the canonical Noether coefficients, given in Appendix A as (3.188) to (3.191).

Therefore we now have the most general "improvement" of the canonical Noether tensor in (3.96), with all possible superpotentials (3.95) and all possible terms proportional to the equations of motion that can be added. These were directly derived from (3.86), therefore we have a direct connection between any linearized gravity energy-momentum tensor, and all which can be obtained by ad-hoc improving the canonical Noether expression in (3.78). By solving the system of linear equations in Appendix A, one finds solutions which satisfy both criteria, that the conserved energy-momentum tensor in (3.86) will be derivable from (3.78) supplemented by the divergence of a superpotential and terms proportional to the equations of motion in (3.77) and (3.87), as given in (3.96). This leads us to our first result: there are infinitely many solutions to the linear system in Appendix A. In other words, there are infinitely many divergences of superpotentials and terms proportional to the equations of motion that can be added to the canonical Noether energy-momentum tensor (3.78) in order to obtain on-shell conserved tensors for linearized gravity. If in addition we use the symmetry conditions in (3.93) and (3.94) we find that there are infinitely many $T^{\rho\sigma}$ which are both symmetric and conserved. The "improvements" used to obtain e.g. the Hilbert and Landau-Liftshitz energy-momentum tensors in (3.80) and (3.83) from $T_C^{\rho\sigma}$ are not special or unique; they are just two of infinitely many possibly solutions. This suggests claims of a meaningful connection of a given $T^{\rho\sigma}$ to Noether's first theorem using the ad-hoc "improvement" method should not be made.

To recap, we will summarize the equations in Appendix A that give the conditions necessary for linearized gravity energy-momentum tensors of the form (3.96). Equations (3.136) to (3.138) give the conditions necessary for the most general linearized gravity energy-momentum tensor (3.86) to be expressed as the canonical Noether energy-momentum tensor improved by the divergence of a superpotential and terms proportional to the equations of motion. Equations (3.139) to (3.147) are the conditions for coefficient splitting modified by the canonical

Noether tensor. Equations (3.148) to (3.159) are the conditions required to have a superpotential antisymmetric in $[\rho\alpha]$. Equations (3.160) to (3.187) are the conditions for each of the 6 ζ_n equations of motion, and (3.188) to (3.191) are the conditions on the M_n terms within. Finally if we wish to derive symmetric expressions we can use the symmetry conditions (3.93) and (3.94) from earlier in the article.

3.3.4 Verifying the general system for the Hilbert and Landau-Lifshitz energy-momentum tensors

We will now use our motivating examples (Hilbert and Landau-Lifshitz energy-momentum tensors) to apply the general results.

Hilbert coefficients and solution

For the Hilbert coefficients in (3.79), from (3.86) we have $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$, $c_4 = \frac{1}{2}$, $a_1 = \frac{1}{2}$, $b_{11_i} = -1$, $b_{11_{ii}} = -1$, $b_4 = 1$, $b_5 = 1$, $b_1 = -1$, $b_2 = -\frac{1}{2}$, $b_6 = -\frac{1}{2}$, $b_7 = \frac{1}{2}$, $b_{9_{ii}} = \frac{1}{2}$, $b_{9_i} = \frac{1}{2}$, $d_{6_i} = 1$, $d_{6_{ii}} = 1$, $d_2 = -1$, $d_4 = -1$, $d_1 = \frac{1}{2}$, $d_{9_i} = -\frac{1}{2}$, and $d_{9_{ii}} = -\frac{1}{2}$. These satisfy the symmetry conditions (3.93) and (3.94).

This is a solution to the linear system in Appendix A, with $\zeta_4 = -2$, and all other $\zeta_n = 0$. These coefficients fix the general superpotential in (3.95) to yield $\Psi^{[\rho\alpha]\sigma} = \frac{1}{2}h^{\rho\sigma}\partial^\alpha h - \frac{1}{2}h^{\sigma\alpha}\partial^\rho h - h^{\alpha\beta}\partial_\beta h^{\rho\sigma} + h^{\rho\beta}\partial_\beta h^{\sigma\alpha} + h^\sigma{}_\beta\partial^\alpha h^{\rho\beta} - h^\sigma{}_\beta\partial^\rho h^{\alpha\beta} - \frac{1}{2}\eta^{\sigma\alpha}h^{\rho\omega}\partial_\omega h + \frac{1}{2}\eta^{\rho\sigma}h^{\alpha\beta}\partial_\beta h$. This is exactly the well known Hilbert superpotential in (3.81). Substituting these solutions back into (3.96) we immediately obtain the well-known result for the Hilbert energy-momentum tensor (3.80). Therefore the general system of equations in Appendix A recovers the Hilbert result.

Landau-Lifshitz coefficients and solution

For the Landau-Lifshitz coefficients in (3.82), from (3.86) we have $c_1 = \frac{3}{4}$, $c_2 = -\frac{1}{4}$, $c_4 = \frac{1}{2}$, $c_5 = -1$, $b_6 = -1$, $b_2 = -\frac{3}{2}$, $b_1 = 1$, $b_3 = -1$, $b_7 = \frac{1}{2}$, $b_4 = 1$, $b_{9_{ii}} = 1$, $b_{9_i} = 1$, $b_{8_i} = \frac{1}{2}$, $b_{8_{ii}} = \frac{1}{2}$, $b_{11_i} = -1$, and $b_{11_{ii}} = -1$. These satisfy the symmetry conditions (3.93) and (3.94).

This is a solution to the linear system in Appendix A, with $\zeta_1 = 1$, $\zeta_4 = -2$, and all other $\zeta_n = 0$. These coefficients fix the general superpotential in (3.95) to yield $\Psi^{[\rho\alpha]\sigma} = -h^{\rho\sigma}\partial^\alpha h + h^{\sigma\alpha}\partial^\rho h - \frac{1}{2}h\partial^\alpha h^{\rho\sigma} + \frac{1}{2}h\partial^\rho h^{\sigma\alpha} + h^{\rho\sigma}\partial_\beta h^{\alpha\beta} - h^{\sigma\alpha}\partial_\beta h^{\rho\beta} + h^\sigma{}_\beta\partial^\alpha h^{\rho\beta} - h^\sigma{}_\beta\partial^\rho h^{\alpha\beta} - \frac{1}{2}\eta^{\sigma\alpha}h\partial^\rho h + \frac{1}{2}\eta^{\rho\sigma}h\partial^\alpha h - \frac{1}{2}\eta^{\rho\sigma}h\partial_\omega h^{\omega\alpha} + \frac{1}{2}\eta^{\sigma\alpha}h\partial_\omega h^{\rho\omega}$. This is exactly the well known Landau-Lifshitz superpotential (3.84). Substituting these solutions back into (3.96) we immediately obtain the well known result for the Landau-Lifshitz energy-momentum tensor in (3.83). Therefore the general system of equations in Appendix A recovers the Landau-Lifshitz result.

3.3.5 Two new energy-momentum tensors derivable from ad-hoc improving the canonical Noether energy-momentum tensor

We now derive two new energy-momentum tensors[1] that can be obtained from improving the canonical Noether tensor by solving the system of equations in Appendix A, just like the Hilbert (3.79) and Landau-Lifshitz (3.82) expressions.

Elizabeth energy-momentum tensor

For the Elizabeth energy-momentum tensor $T_E^{\rho\sigma}$, we will use the symmetry conditions (3.93) and (3.94). We find a symmetric solution to the linear system in Appendix A to be $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$, $c_4 = \frac{1}{2}$, $b_1 = 1$, $b_2 = -\frac{1}{2}$, $b_3 = -1$, $b_4 = 1$, $b_6 = -\frac{1}{2}$, $b_7 = \frac{1}{2}$, $b_9 = \frac{1}{2}$, $b_{11} = -1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$, $a_4 = \frac{1}{2}$, $d_1 = \frac{1}{2}$, $d_9 = -\frac{1}{2}$. This yields from (3.86) the energy-momentum tensor,

$$\begin{aligned}T_E^{\rho\sigma} = {} & \partial_\alpha h^{\rho\sigma}\partial_\beta h^{\alpha\beta} - \frac{1}{2}\partial_\alpha h^{\rho\sigma}\partial^\alpha h - \partial_\alpha h^{\rho\alpha}\partial_\beta h^{\sigma\beta} + \partial_\alpha h^\rho{}_\beta \partial^\alpha h^{\sigma\beta} \\ & - \frac{1}{2}\partial^\rho h \partial^\sigma h + \frac{1}{2}\partial^\rho h_{\alpha\beta}\partial^\sigma h^{\alpha\beta} + \frac{1}{2}\partial^\rho h\partial_\alpha h^{\sigma\alpha} + \frac{1}{2}\partial^\sigma h\partial_\alpha h^{\rho\alpha} - \partial^\rho h_{\alpha\beta}\partial^\alpha h^{\sigma\beta} - \partial^\sigma h_{\alpha\beta}\partial^\alpha h^{\rho\beta} \\ & + \frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + \frac{1}{2}\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} \\ & + \frac{1}{2}h^{\rho\sigma}\partial_\alpha\partial^\alpha h - \frac{1}{2}h^{\rho\alpha}\partial^\sigma\partial_\alpha h - \frac{1}{2}h^{\sigma\alpha}\partial^\rho\partial_\alpha h + \frac{1}{2}\eta^{\rho\sigma}h_{\alpha\beta}\partial^\alpha\partial^\beta h - \frac{1}{2}\eta^{\rho\sigma}h\partial_\alpha\partial_\beta h^{\alpha\beta} + \frac{1}{2}\eta^{\rho\sigma}h\partial_\alpha\partial^\alpha h.\end{aligned} \tag{3.97}$$

This fixes $\zeta_4 = -2$ and $\zeta_5 = -\frac{1}{2}$ in (3.96), with the remaining $\zeta_n = 0$. From (3.95) we obtain the superpotential,

$$\begin{aligned}\Psi_E^{[\rho\alpha]\sigma} = {} & -\frac{1}{2}h^{\rho\sigma}\partial^\alpha h + \frac{1}{2}h^{\sigma\alpha}\partial^\rho h + h^{\rho\sigma}\partial_\beta h^{\alpha\beta} - h^{\sigma\alpha}\partial_\beta h^{\rho\beta} \\ & + h^\sigma{}_\beta\partial^\alpha h^{\rho\beta} - h^\sigma{}_\beta\partial^\rho h^{\alpha\beta} - \frac{1}{2}\eta^{\sigma\alpha}h^{\rho\omega}\partial_\omega h + \frac{1}{2}\eta^{\rho\sigma}h^{\alpha\beta}\partial_\beta h.\end{aligned} \tag{3.98}$$

Thus (3.97) can be derived from the canonical Noether energy-momentum tensor (3.78) by adding ad-hoc $\partial_\alpha\Psi_E^{[\rho\alpha]\sigma}$ and $-2h^\sigma_\alpha E^{\rho\alpha} - \frac{1}{2}\eta^{\rho\sigma}h\mathbf{E}$,

$$T_E^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha\Psi_E^{[\rho\alpha]\sigma} - 2h^\sigma_\alpha E^{\rho\alpha} - \frac{1}{2}\eta^{\rho\sigma}h\mathbf{E}. \tag{3.99}$$

[1] We will call the new expressions the Audrey and Elizabeth energy-momentum tensors, named after our Grandmothers.

Similarly, we can obtain infinitely many conserved, symmetric energy-momentum tensors for linearized gravity from the canonical Noether energy-momentum tensor (two of which are the Hilbert (3.79) and Landau-Lifshitz (3.82) expressions).

Audrey energy-momentum tensor

For the Audrey energy-momentum tensor $T_A^{\rho\sigma}$, we will use the conditions $a_n = 0$ and $b_n = 0$. Using these conditions we will prove that no symmetric expressions exist without these terms. However, a conserved expression can be derived by improving the canonical Noether energy-momentum tensor (3.78). We find a solution to the coefficients in Appendix A with $a_n = 0$ and $b_n = 0$ to be $c_1 = -\frac{1}{4}$, $c_2 = \frac{1}{4}$, $c_3 = -1$, $c_4 = \frac{1}{2}$, $c_5 = \frac{1}{2}$, $d_8 = -\frac{1}{2}$. $d_{12_{ii}} = -1$, $d_7 = \frac{1}{2}$, $d_{11_{ii}} = -\frac{1}{2}$, $d_{9_i} = -\frac{1}{2}$ and $d_{10_i} = 2$. This yields from (3.86) the energy-momentum tensor,

$$\begin{aligned} T_A^{\rho\sigma} &= -\frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h \partial^\alpha h + \frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} - \eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\lambda h^\lambda{}_\beta + \frac{1}{2}\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} + \frac{1}{2}\eta^{\rho\sigma}\partial_\alpha h^{\alpha\beta}\partial_\beta h \\ &+ \frac{1}{2}h\partial^\rho\partial^\sigma h - \frac{1}{2}h_{\alpha\beta}\partial^\rho\partial^\sigma h^{\alpha\beta} - \frac{1}{2}h^{\rho\alpha}\partial^\sigma\partial_\alpha h - \frac{1}{2}h\partial^\sigma\partial_\alpha h^{\rho\alpha} - h_{\alpha\beta}\partial^\sigma\partial^\alpha h^{\rho\beta} + 2h^{\rho\alpha}\partial^\sigma\partial^\beta h_{\alpha\beta}. \end{aligned} \tag{3.100}$$

This fixes $\zeta_6 = -1$ in (3.96), with the remaining $\zeta_n = 0$. From (3.95) we obtain the superpotential,

$$\begin{aligned} \Psi_A^{[\rho\alpha]\sigma} &= -h^\alpha{}_\beta\partial^\sigma h^{\rho\beta} + h^\rho{}_\beta\partial^\sigma h^{\alpha\beta} + \frac{1}{2}\eta^{\sigma\alpha}h\partial^\rho h - \frac{1}{2}\eta^{\rho\sigma}h\partial^\alpha h - \frac{1}{2}\eta^{\sigma\alpha}h^{\omega\beta}\partial^\rho h_{\omega\beta} + \frac{1}{2}\eta^{\rho\sigma}h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} \\ &- \frac{1}{2}\eta^{\sigma\alpha}h^{\rho\omega}\partial_\omega h + \frac{1}{2}\eta^{\rho\sigma}h^{\alpha\beta}\partial_\beta h + \frac{1}{2}\eta^{\rho\sigma}h\partial_\omega h^{\omega\alpha} - \frac{1}{2}\eta^{\sigma\alpha}h\partial_\omega h^{\rho\omega} + \eta^{\sigma\alpha}h^{\rho\omega}\partial^\beta h_{\omega\beta} - \eta^{\rho\sigma}h^{\alpha\beta}\partial_\lambda h^\lambda{}_\beta. \end{aligned} \tag{3.101}$$

Thus (3.100) can be derived from the canonical Noether energy-momentum tensor (3.78) by adding ad-hoc $\partial_\alpha\Psi_A^{[\rho\alpha]\sigma}$ and $-\eta^{\rho\sigma}h_{\alpha\beta}E^{\alpha\beta}$,

$$T_A^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha\Psi_A^{[\rho\alpha]\sigma} - \eta^{\rho\sigma}h_{\alpha\beta}E^{\alpha\beta}. \tag{3.102}$$

We cannot have a symmetric expression here because $a_n = 0$ and $b_n = 0$ fixes $d_{12_i} = 0$ and $d_{12_{ii}} = -1$ which breaks the symmetry conditions (3.93) and (3.94). Both conditions must hold in order to have a symmetric energy-momentum tensor in (3.86).

3.3.6 There exist infinitely many symmetric, conserved energy-momentum tensors from the Belinfante superpotential alone

We now present, perhaps our most significant result, that the Belinfante superpotential is associated with infinitely many symmetric and conserved linearized gravity energy-momentum tensors. This is an important result because despite the various possible superpotentials one can add (such as $\Psi_{LL}^{[\rho\alpha]\sigma}$ or those found in [28]), the Belinfante superpotential is the most commonly published expression. Our result is contrary to popular belief in the recent literature that the Hilbert energy-momentum tensor uniquely specifies the Belinfante energy-momentum tensor [63, 16]. This point is central to the recent Padmanabhan-Deser debate [163, 63], in which the authors have argued about whether or not general relativity can be derived from spin-2 using a $T^{\mu\nu}$ resulting from Noether's theorem. Deser claimed that ad-hoc improving the canonical Noether tensor (3.78) with the Belinfante superpotential uniquely gives the Hilbert energy-momentum tensor (3.79), thus he argued one does not have to use Noether's theorem at all to have a result from Noether's theorem, they can simply use the Hilbert approach. Such claims of general equivalence of the Noether and Hilbert methods for deriving an energy-momentum tensor has since been disproved in [13]. Deser's assertions come from results that have a long history [175, 95, 6, 32, 176, 76, 134, 133, 171] of investigating the relationship between the Belinfante and Hilbert energy-momentum tensors. The general conclusion in the literature is that one can add the divergence of the Belinfante superpotential and terms proportional to the equation of motion to reconcile the Belinfante and Hilbert definitions (this was confirmed by our results). But this does not prove uniqueness! Indeed the Belinfante superpotential coincides with what we found for Hilbert in (3.81). However, as we will prove, this result is not unique because there are infinitely many symmetric and conserved expressions associated to this particular superpotential alone; infinitely many combinations of the equations of motion in (3.96) are solutions to the system of equations in Appendix A when the Belinfante superpotential is fixed. Therefore one cannot make the claim that the Hilbert energy-momentum tensor is uniquely specified by ad-hoc adding the divergence of Belinfante superpotential and terms proportional to the equations of motion. In other words, no significant connection exists between Noether's first theorem and the Hilbert energy-momentum tensor in spin-2 linearized gravity, as supported by the recent disproof in [13].

The Belinfante superpotential for spin-2 Fierz-Pauli theory

We will start by recapping the Belinfante superpotential derivation and showing that it matches the superpotential obtained from the Hilbert energy-momentum tensor in (3.79). The Belinfante improvement procedure consists of adding the divergence of a superpotential $b^{[\mu\alpha]\nu}$ to the canonical Noether energy-momentum tensor. By adding this "improvement" term ($\partial_\alpha b^{[\mu\alpha]\nu}$) one obtains the Belinfante energy-momentum tensor [23],

$$T_B^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha b^{[\rho\alpha]\sigma}. \tag{3.103}$$

The superpotential $b^{[\rho\gamma]\sigma}$ is specifically a combination of the canonical spin angular momentum tensors $S^{\rho[\sigma\gamma]}$ [23],

$$b^{[\rho\gamma]\sigma} = \frac{1}{2}(-S^{\rho[\sigma\gamma]} + S^{\gamma[\sigma\rho]} + S^{\sigma[\gamma\rho]}), \tag{3.104}$$

a result Belinfante attributes to a Dr. Podolansky (without reference) in his article. The $S^{\rho[\sigma\gamma]}$ are sometimes referred to as the spin contributions. The Belinfante prescription for gravity models has been worked out in [7]. The spin angular momentum connection for a second rank $h_{\mu\nu}$ is given by [134],

$$S^{\sigma[\rho\gamma]} = \frac{\partial\mathcal{L}}{\partial\partial_\sigma h^{\mu\nu}}[\eta^{\rho\mu}h^{\gamma\nu} - \eta^{\gamma\mu}h^{\rho\nu} + \eta^{\rho\nu}h^{\gamma\mu} - \eta^{\gamma\nu}h^{\rho\mu}]. \tag{3.105}$$

Thus we require the derivative of the Fierz-Pauli Lagrangian density $\frac{\partial\mathcal{L}_{\mathcal{FP}}}{\partial(\partial_\sigma h^{\mu\nu})}$ in (3.76). Substituting $\frac{\partial\mathcal{L}_{\mathcal{FP}}}{\partial(\partial_\sigma h^{\mu\nu})}$ into (3.105), we have for the Belinfante superpotential in (3.104),

$$b^{[\rho\gamma]\sigma} = -\frac{1}{2}[\eta^{\gamma\alpha}h^{\sigma\rho}-\eta^{\rho\alpha}h^{\gamma\sigma}]\partial_\alpha h^\zeta_\zeta-\frac{1}{2}[\eta^{\sigma\gamma}h^{\rho\nu}-\eta^{\sigma\rho}h^{\gamma\nu}]\partial_\nu h^\zeta_\zeta-[\eta^{\rho\alpha}h^{\gamma\nu}-\eta^{\gamma\alpha}h^{\rho\nu}]\partial_\nu h^\sigma_\alpha-h^{\sigma\nu}[\partial^\rho h^\gamma_{\ \nu}-\partial^\gamma h^\rho_\nu], \tag{3.106}$$

which is exactly what we found for the Hilbert energy-momentum tensor in (3.81). Thus we have the well known result,

$$b^{[\rho\gamma]\sigma} = \Psi_H^{[\rho\gamma]\sigma}. \tag{3.107}$$

However, the Belinfante superpotential is not enough by itself to specify the Hilbert tensor (3.80),

$$T_H^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\gamma b^{[\rho\gamma]\sigma} - 2h^\sigma_\beta E^{\rho\beta}. \tag{3.108}$$

We also need the very specific $-2h^\sigma_\beta E^{\rho\beta}$ piece to reconcile the two results; this does not prove uniqueness of $T_H^{\rho\sigma}$ for the Belinfante superpotential. Fixing coefficients in (3.86) such that the only solutions in Appendix A are those with the specific Belinfante superpotential in (3.106), we will see that the Belinfante superpotential alone yields infinitely many possible results, of which one happens to be the Hilbert expression.

There are infinitely many solutions following from the ad-hoc addition of the divergence of the Belinfante superpotential

We now prove our main result, that there are infinitely solutions even when fixing the Belinfante superpotential. If we fix (3.96) with the Belinfante superpotential (3.106) we have,

$$T^{\rho\sigma} = T_C^{\rho\sigma} + \partial_\alpha b^{[\rho\alpha]\sigma} + \zeta_1 h E^{\rho\sigma} + \zeta_2 h^\rho_\alpha E^{\sigma\alpha} + \zeta_3 h^{\rho\sigma}\mathbf{E} + \zeta_4 h^\sigma_\alpha E^{\rho\alpha} + \zeta_5 \eta^{\rho\sigma} h\mathbf{E} + \zeta_6 \eta^{\rho\sigma} h_{\alpha\beta} E^{\alpha\beta}. \quad (3.109)$$

Therefore if we improve the canonical Noether expression with the divergence of the Belinfante superpotential, we in theory can have the six addition equation of motion pieces. However fixing the Belinfante superpotential coefficients in highly restrictive on the linear system in Appendix A. In particular the superpotential conditions in (3.148) to (3.159) are much more restricted as now:

$$B_2 + \bar{D}_1 - \bar{D}_3 = -\frac{1}{2} \quad (3.110)$$
$$b_{9_i} = \frac{1}{2} \quad (3.111)$$
$$\bar{B}_2 - \bar{D}_1 + \bar{D}_3 = 0 \quad (3.112)$$
$$b_{8_i} = 0 \quad (3.113)$$
$$b_1 - d_4 = 0 \quad (3.114)$$
$$b_3 = 0 \quad (3.115)$$
$$d_4 = -1 \quad (3.116)$$
$$b_5 = 1 \quad (3.117)$$
$$d_{12_i} = 0 \quad (3.118)$$
$$B_4 + \bar{D}_{5_i} - \bar{D}_{5_{ii}} = 0 \quad (3.119)$$
$$d_{12_{ii}} = 0 \quad (3.120)$$
$$b_{11_{ii}} + 1 = 0 \quad (3.121)$$
$$\bar{B}_4 - \bar{D}_{5_i} + \bar{D}_{5_{ii}} = 1 \quad (3.122)$$
$$b_{11_i} = -1 \quad (3.123)$$
$$b_6 + \frac{1}{2} = 0 \quad (3.124)$$
$$c_1 - \frac{1}{4} = 0 \quad (3.125)$$
$$d_8 = 0 \quad (3.126)$$
$$c_2 + \frac{1}{4} = 0 \quad (3.127)$$
$$b_{8_{ii}} - \frac{1}{2} = -\frac{1}{2} \quad (3.128)$$
$$C_5 + \bar{A}_1 - \bar{A}_2 = \frac{1}{2} \quad (3.129)$$
$$\bar{C}_5 - \bar{A}_1 + \bar{A}_2 = 0 \quad (3.130)$$
$$b_{9_{ii}} - \frac{1}{2} = 0 \quad (3.131)$$
$$b_{10_{ii}} - d_{12_{ii}} = 0 \quad (3.132)$$
$$c_3 = 0 \quad (3.133)$$

Using Appendix A with these superpotential conditions, and the symmetry conditions in (3.93) and (3.94), we find a solution with 3 free parameters ζ_1, ζ_3 and ζ_5. The other ζ_n are $\zeta_2 = 0$, $\zeta_4 = -2$ and $\zeta_6 = 0$. This solution is $a_2 = \frac{1}{2}\zeta_1 + \zeta_5$, $a_4 = -\frac{1}{2}\zeta_1 - \zeta_5$ $b_1 = -1$, $b_2 = -\frac{1}{2}$,

$b_4 = 1$, $b_5 = 1$, $b_6 = -\frac{1}{2}$, $b_7 = \frac{1}{2}$, $b_{9_i} = \frac{1}{2}$, $b_{9_{ii}} = \frac{1}{2}$, $b_{11_i} = -1$, $b_{11_{ii}} = -1$, $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$, $c_4 = \frac{1}{2}$ $d_1 = \frac{1}{2} - \zeta_3$, $d_2 = -1 + \zeta_3$, $d_3 = \frac{1}{2}\zeta_1$, $d_4 = -1$, $d_{6_i} = 1$, $d_{6_{ii}} = 1$, $d_7 = \frac{1}{2}\zeta_1$, $d_{9_i} = -\frac{1}{2}$, $d_{9_{ii}} = -\frac{1}{2}$, $d_{11_i} = -\frac{1}{2}$, $d_{11_{ii}} = -\frac{1}{2}$. Using (3.86) we have infinitely many conserved and symmetric energy-momentum tensors $T^{\rho\sigma}_{IB}$,

$$
\begin{aligned}
T^{\rho\sigma}_{IB} = &-\partial_\alpha h^{\rho\sigma}\partial_\beta h^{\alpha\beta} - \frac{1}{2}\partial_\alpha h^{\rho\sigma}\partial^\alpha h + \partial_\alpha h^{\rho}{}_{\beta}\partial^\alpha h^{\sigma\beta} + \partial_\alpha h^{\rho\beta}\partial_\beta h^{\sigma\alpha} \\
&- \frac{1}{2}\partial^\rho h\partial^\sigma h + \frac{1}{2}\partial^\rho h_{\alpha\beta}\partial^\sigma h^{\alpha\beta} + \frac{1}{2}\partial^\rho h\partial_\alpha h^{\sigma\alpha} + \frac{1}{2}\partial^\sigma h\partial_\alpha h^{\rho\alpha} \\
&- \partial^\rho h_{\alpha\beta}\partial^\alpha h^{\sigma\beta} - \partial^\sigma h_{\alpha\beta}\partial^\alpha h^{\rho\beta} + \frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\eta^{\rho\sigma}\partial_\alpha h_{\beta\lambda}\partial^\alpha h^{\beta\lambda} + \frac{1}{2}\eta^{\rho\sigma}\partial_\alpha h_{\lambda\beta}\partial^\lambda h^{\alpha\beta} \\
&+ (\frac{1}{2} - \zeta_3)h^{\rho\sigma}\partial_\alpha\partial^\alpha h + (-1 + \zeta_3)h^{\rho\sigma}\partial_\alpha\partial_\beta h^{\alpha\beta} + \frac{1}{2}\zeta_1 h\partial_\alpha\partial^\alpha h^{\rho\sigma} - h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\rho\sigma} \\
&+ h^{\rho\alpha}\partial_\alpha\partial_\beta h^{\sigma\beta} + h^{\sigma\alpha}\partial_\alpha\partial_\beta h^{\rho\beta} + \frac{1}{2}\zeta_1 h\partial^\rho\partial^\sigma h - \frac{1}{2}h^{\rho\alpha}\partial^\sigma\partial_\alpha h - \frac{1}{2}h^{\sigma\alpha}\partial^\rho\partial_\alpha h - \frac{1}{2}h\partial^\rho\partial_\alpha h^{\sigma\alpha} - \frac{1}{2}h\partial^\sigma\partial_\alpha h^{\rho\alpha} \\
&+ (\frac{1}{2}\zeta_1 + \zeta_5)\eta^{\rho\sigma}h\partial_\alpha\partial_\beta h^{\alpha\beta} + (-\frac{1}{2}\zeta_1 - \zeta_5)\eta^{\rho\sigma}h\partial_\alpha\partial^\alpha h, \quad (3.134)
\end{aligned}
$$

where the subscript IB denotes the 'infinite Belinfante' expressions, since using (3.96) we have infinitely many symmetric, conserved spin-2 energy-momentum tensor that can be derived from ad-hoc addition of the Belinfante superpotential to the canonical Noether energy-momentum tensor,

$$T^{\rho\sigma}_{IB} = T^{\rho\sigma}_{C} + \partial_\alpha b^{[\rho\alpha]\sigma} + \zeta_1 hE^{\rho\sigma} + \zeta_3 h^{\rho\sigma}\mathbf{E} - 2h^{\sigma}_{\alpha}E^{\rho\alpha} + \zeta_5\eta^{\rho\sigma}h\mathbf{E}. \quad (3.135)$$

Therefore we have proven that adding the "improvement" terms associated to the Belinfante superpotential does not specify a unique result; no meaningful connection to Noether's first theorem can be claimed by specifying Belinfante "improvements". Note that we solved for particular free coefficients such that we trivially recover the Hilbert energy-momentum tensor (3.79) when the free coefficients ζ_1, ζ_3 and ζ_5 are set to zero.

3.3.7 Summary and Discussion

In this article, we considered the most general possible linearized gravity energy-momentum tensor using a procedure developed by Fock [75, 109] and recently applied in a more restricted case to the non-uniqueness problem in linearized gravity by Bičák and Schmidt [28]. Using this general expression we derived the most general possible superpotential and terms proportional to the equations of motion (3.88) and used this expression to derive the most general possible "improvements" (3.96) of the canonical Noether energy-momentum tensor of spin-2 Fierz-Pauli theory (3.78). In Appendix A we gave the most general linear system of equations that represents all such solutions to (3.96). In addition conditions (3.93) and (3.94) can be imposed to guarantee symmetry ($T^{\mu\nu} = T^{\nu\mu}$) of the solution.

Solving this general system in (3.96) and Appendix A we have proven several results related to the ad-hoc "improvement" of energy-momentum tensors in linearized gravity. The addition of a superpotential and terms proportional to the equations of motion to the canonical Noether energy-momentum tensor is often presented as a method for obtaining various energy-momentum tensors from Noether's first theorem, such as the Hilbert (3.79) and Landau-Lifshitz (3.82) expressions in linearized gravity. We have shown that these ad-hoc "improvements" do not provide a unique and/or meaningful connection to Noether's first theorem. To highlight this point we derived two new energy-momentum tensors, the Audrey (3.100) and Elizabeth (3.97) energy-momentum tensors. The Elizabeth energy-momentum tensor gives a symmetric expression connected to the canonical Noether tensor in the same way as the Hilbert and Landau-Lifshitz energy-momentum tensors. The Audrey energy-momentum tensor gives a non-symmetric expression, and proves that no symmetric expression can be built when conditions $a_n = 0$ and $b_n = 0$ are imposed on (3.86). Finally we prove that there are infinitely many symmetric and conserved energy-momentum tensors associated to the Belinfante superpotential, one of which is the Hilbert energy-momentum tensors. This is contrary to the conventional wisdom that this association, given in (3.108), is unique for linearized gravity.

Our results show that there is no unique or meaningful connection between the canonical Noether energy-momentum tensor and any expression obtained after ad-hoc adding the divergence of superpotentials and terms proportional to the equations of motion (i.e. any expressions not derived directly from Noether's first theorem). There are infinitely many such results of this form, and infinitely many even if we restrict our attention to the Belinfante superpotential alone. Selecting a unique energy-momentum tensor for linearized gravity is of course difficult because none are invariant under the spin-2 gauge transformation (linearized diffeomorphisms) as shown by Magnano and Sokolowski [140]. What is for certain, however, is that outside of the canonical Noether expression, any connection to Noether's first theorem of the various energy-momentum tensors in the literature should be revisited. This is especially true given the recent proof that the Noether and Hilbert energy-momentum tensors are not, in general, equivalent [13]. The question still remains as to what is the physical significance of the many published spin-2 gravitational energy-momentum tensors in the literature, as well as in general relativity [190, 164, 152, 25], and gravity theories as a whole [169, 123, 154, 178]. Many of the linearized gravity energy-momentum tensors were highlighted by Bičák and Schmidt [28], a study which in part stemmed from continued research in linearized gravity by Butcher et al.

in the recent literature [41, 40, 42, 43]. In electrodynamics, fundamental equations such as the Lorentz force law and Poynting's theorem are expressed through the uniquely defined physical energy-momentum tensor. In spin-2 Fierz-Pauli theory, writing down analogous laws requires a unique energy-momentum tensor, for which there is still no consensus on which to choose. In addition, the self coupling problem of $h_{\mu\nu}T^{\mu\nu}$ in the Padmanabhan-Deser debate [163, 63] requires a specific expression which the authors could not agree on, further emphasizing the need for a uniquely defined expression. The linearized Landau-Lifshitz energy-momentum pseudotensor has been used to model observations in one (the Hulse-Taylor) binary pulsar system [107, 187, 166, 131]. Others make claims in support of other expressions, such as the Hilbert (metric) energy-momentum tensor in Minkowski spacetime to be the truly physical energy-momentum tensor. For these numerous other linearized gravity energy-momentum tensors in the physics literature, however, experimental or observational verification cannot easily be found.

Due to Magnano and Sokolowski's no-go result [140], one can consider energy-momentum tensors in higher derivative gravity in order to obtain a spin-2 gauge invariant expression (invariant under linearized diffeomorphisms), such as the variants of the Bel-Robinson tensor [3, 92], or the linearized Gauss-Bonnet gravity energy-momentum tensor [167, 14, 9], which are both invariant under the spin-2 gauge transformation (linearized diffeomorphisms). However, since these models require higher derivative actions, they are not connected to spin-2 Fierz-Pauli theory via standard Lagrangian based energy-momentum derivations such as the Noether method or the Hilbert (metric) method. We note that additional insight about linearized gravity can be found through the Hamiltonian approach [97, 51, 179, 189]. Research on gravitational waves from the linearized gravity equations has continued in recent decades [144]; interests that have only been increasing since the LIGO results in 2016 [1]. In electrodynamics, the radiation equations are developed from the unique energy-momentum tensor of the theory, emphasizing the need to sort out the non-uniqueness problem in linearized gravity. Various energy-momentum tensors have been proposed to model gravitational radiation, such as the Bel-Robinson tensor [62, 94], yet only the linearized Landau-Lifshitz energy-momentum pseudotensor has the observational evidence associated to the Hulse-Taylor binary [107, 187, 166, 131]. Sorting out which of the many published expressions correspond to physical phenomena is a fundamental problem which can give great insight into the theoretical framework of gravitational energy. With many published gravitational energy-momentum tensors in the literature, it is not clear which (if any, see philosophical debates on this topic [105, 173]) to use to write down a unique set of physical conservation laws for linearized gravity. We hope that our results will help further progress in this direction, and to clarify the relationship of the many published expressions to the canonical Noether energy-momentum tensor.

3.3.8 Appendix A - General energy-momentum tensor system of linear equations

The following Appendix includes the system of linear equations corresponding to the most general energy-momentum tensor for linearized gravity (3.86) under the condition that the most general expression can be derived from the canonical Noether tensor supplemented by the ad-hoc addition of the divergence of a superpotential and terms proportional to the equations of motion given in (3.96). In addition to the equations given in this Appendix, (3.93) and (3.94) can be imposed to derive a symmetric energy-momentum tensor.

Canonical Noether conditions

$$b_7 - d_8 = \frac{1}{2} \tag{3.136}$$

$$c_4 = \frac{1}{2} \tag{3.137}$$

$$b_{10_i} - d_{12_i} = 0 \tag{3.138}$$

Coefficient splitting conditions

$$a_1 = A_1 + \bar{A}_1 \tag{3.139}$$

$$a_2 = A_2 + \bar{A}_2 \tag{3.140}$$

$$b_2 = B_2 + \bar{B}_2 \tag{3.141}$$

$$b_4 = B_4 + \bar{B}_4 \tag{3.142}$$

$$(c_5 + \frac{1}{2}) = C_5 + \bar{C}_5 \tag{3.143}$$

$$d_1 = D_1 + \bar{D}_1 \tag{3.144}$$

$$d_3 = D_3 + \bar{D}_3 \tag{3.145}$$

$$d_{5_i} = D_{5_i} + \bar{D}_{5_i} \tag{3.146}$$

$$d_{5_{ii}} = D_{5_{ii}} + \bar{D}_{5_{ii}} \tag{3.147}$$

Superpotential conditions

$$B_2 + \bar{D}_1 - \bar{D}_3 + b_{9_i} = 0 \tag{3.148}$$
$$\bar{B}_2 - \bar{D}_1 + \bar{D}_3 + b_{8_i} = 0 \tag{3.149}$$
$$b_1 - d_4 + b_3 = 0 = 0 \tag{3.150}$$
$$C_5 + \bar{A}_1 - \bar{A}_2 + (b_{8_{ii}} - \frac{1}{2}) = 0 \tag{3.151}$$
$$\bar{C}_5 - \bar{A}_1 + \bar{A}_2 + (b_{9_{ii}} - \frac{1}{2}) = 0 \tag{3.152}$$
$$c_3 + b_{10_{ii}} - d_{12_{ii}} = 0 \tag{3.153}$$
$$d_4 + b_5 = 0 \tag{3.154}$$
$$d_{12_i} + B_4 + \bar{D}_{5_i} - \bar{D}_{5_{ii}} = 0 \tag{3.155}$$
$$d_{12_{ii}} + (b_{11_{ii}} + 1) = 0 \tag{3.156}$$
$$\bar{B}_4 - \bar{D}_{5_i} + \bar{D}_{5_{ii}} + b_{11_i} = 0 \tag{3.157}$$
$$(c_1 - \frac{1}{4}) + (b_6 + \frac{1}{2}) = 0 \tag{3.158}$$
$$(c_2 + \frac{1}{4}) + d_8 = 0 \tag{3.159}$$

Linear system of equations for the equations of motion

$$-\frac{1}{2}\zeta_1 = \bar{M}_4 \tag{3.160}$$
$$\frac{1}{2}\zeta_1 = D_3 - \bar{B}_2 + \bar{D}_1 \tag{3.161}$$
$$\frac{1}{2}\zeta_1 = d_7 - (b_6 + \frac{1}{2}) \tag{3.162}$$
$$-\frac{1}{2}\zeta_1 = d_{11_i} - b_{8_i} \tag{3.163}$$
$$-\frac{1}{2}\zeta_1 = d_{11_{ii}} - (b_{9_{ii}} - \frac{1}{2}) \tag{3.164}$$
$$\frac{1}{2}\zeta_1 = \bar{M}_3 \tag{3.165}$$
$$-\frac{1}{2}\zeta_2 = \bar{M}_2 \tag{3.166}$$
$$\frac{1}{2}\zeta_2 = D_{5_i} + \bar{D}_{5_{ii}} - B_4 \tag{3.167}$$
$$\frac{1}{2}\zeta_2 = d_{9_i} - (b_{8_{ii}} - \frac{1}{2}) \tag{3.168}$$
$$-\frac{1}{2}\zeta_2 = d_{10_i} + d_{12_{ii}} - b_{10_{ii}} - (b_{11_{ii}} + 1) \tag{3.169}$$
$$-\frac{1}{2}\zeta_2 = d_{6_i} - b_5 \tag{3.170}$$
$$\frac{1}{2}\zeta_2 = \bar{M}_1 \tag{3.171}$$

$$\zeta_3 = M_1 \tag{3.172}$$
$$-\zeta_3 = M_2 \tag{3.173}$$
$$\zeta_5 = M_3 \tag{3.174}$$
$$-\zeta_5 = M_4 \tag{3.175}$$

$$-\frac{1}{2}\zeta_4 = \hat{M}_2 \tag{3.176}$$
$$\frac{1}{2}\zeta_4 = D_{5_{ii}} + \bar{D}_{5_i} - \bar{B}_4 \tag{3.177}$$
$$\frac{1}{2}\zeta_4 = d_{9_{ii}} - b_{9_i} \tag{3.178}$$
$$-\frac{1}{2}\zeta_4 = d_{10_{ii}} + d_{12_i} - b_{10_i} - b_{11_i} \tag{3.179}$$
$$-\frac{1}{2}\zeta_4 = d_{6_{ii}} - b_3 \tag{3.180}$$
$$\frac{1}{2}\zeta_4 = \hat{M}_1 \tag{3.181}$$
$$-\frac{1}{2}\zeta_6 = \hat{M}_4 \tag{3.182}$$
$$\frac{1}{2}\zeta_6 = a_5 - (c_2 + \frac{1}{4}) \tag{3.183}$$
$$\frac{1}{2}\zeta_6 = A_1 - C_5 + \bar{A}_2 \tag{3.184}$$
$$-\frac{1}{2}\zeta_6 = \frac{1}{2}(a_3 - c_3) \tag{3.185}$$
$$-\frac{1}{2}\zeta_6 = \frac{1}{2}(a_3 - c_3) \tag{3.186}$$
$$\frac{1}{2}\zeta_6 = \hat{M}_3 \tag{3.187}$$

$$d_2 + d_4 - b_1 = M_1 + \bar{M}_1 + \hat{M}_1 \tag{3.188}$$
$$D_1 - B_2 + \bar{D}_3 = M_2 + \bar{M}_2 + \hat{M}_2 \tag{3.189}$$
$$A_2 - \bar{C}_5 + \bar{A}_1 = M_3 + \bar{M}_3 + \hat{M}_3 \tag{3.190}$$
$$a_4 - (c_1 - \frac{1}{4}) = M_4 + \bar{M}_4 + \hat{M}_4 \tag{3.191}$$

Chapter 4

Deriving Lagrangian densities from physical requirements

This chapter focuses on the problems outlined in Section 1.5.3 of the introduction; in order to realize the axiomatic approach to field theory proposed in this thesis, a concrete procedure for converting axioms into a set of Lagrangian densities is required. In the first article of this chapter, [14] in Section 4.1, we develop one possible method for a concrete procedure by considering the most general scalars of a particular type (order of derivatives, rank of tensor potential) with free coefficients, and solving for the coefficients such that the axioms are satisfied. Doing this for the $N = M = n$ case under the condition that each is invariant under the spin-n gauge transformations, electrodynamics is the unique result for $N = M = 1$, and linearized Gauss-Bonnet gravity is the unique result for $N = M = 2$. This article is published in the *International Journal of Modern Physics D*. This connection is further explored in the section article [9] in Section 4.2 where the complete dual formulation is analogously developed and generalized to the Maxwell-like higher spin gauge theories. This article is also published in the *International Journal of Modern Physics D*. Finally the more general $N = M \geq 3$ cases are explored which yield unqiuely the contractions of the curvature tensors of higher spin gauge theories for each spin-n. This article is published in the *Canadian Journal of Physics*. The procedure implemented in this chapter provides a first step towards realizing a method which can obtain complete sets of physical Lagrangian densities from a common set of axioms. Two of the three papers in this chapter feature a single co-author, their contribution to each paper is summarized in the Co-Authorship Statement.

4.1 A connection between linearized Gauss-Bonnet gravity and classical electrodynamics

Abstract A connection between linearized Gauss-Bonnet gravity and classical electrodynamics is found by developing a procedure which can be used to derive completely gauge invariant models. The procedure involves building the most general Lagrangian for a particular order of derivatives (N) and rank of tensor potential (M), then solving such that the model is completely gauge invariant (the Lagrangian density, equation of motion and energy-momentum tensor are all gauge invariant). In the case of $N = 1$ order of derivatives and $M = 1$ rank of tensor potential, electrodynamics is uniquely derived from the procedure. In the case of $N = 2$ order of derivatives and $M = 2$ rank of symmetric tensor potential, linearized Gauss-Bonnet gravity is uniquely derived from the procedure. The natural outcome of the models for classical electrodynamics and linearized Gauss-Bonnet gravity from a common set of rules provides an interesting connection between two well explored physical models.

4.1.1 Motivation

Gauge invariance is a common characteristic among field theories of fundamental interactions. Electrodynamics has a unique property with respect to gauge invariance that will be called *complete gauge invariance* in this article. Complete gauge invariance occurs when the Lagrangian density, equation of motion and energy-momentum tensor are all gauge invariant [39]. Complete gauge invariance of the model is possible only when the Lagrangian density is exactly gauge invariant. Consider the Noether identity for a general potential Φ_A [159, 124, 7],

$$\left(\frac{\partial\mathcal{L}}{\partial\Phi_A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)} + \partial_\mu\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)} + \dots\right)\delta\Phi_A$$
$$+ \partial_\mu\left(\eta^{\mu\nu}\mathcal{L}\delta x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\partial_\omega\delta\Phi_A - \left[\partial_\omega\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\omega\Phi_A)}\right]\delta\Phi_A + ...\right) = 0\,. \quad (4.1)$$

This identity is the result of invariance of the action under simultaneous change of coordinates and fields. For a given coordinate change δx_ν and corresponding change of fields $\delta\Phi_A$ of the Lagrangian density, a conservation law will follow from the particular form of $\delta\Phi_A$, such as in the case of Lorentz translation the energy-momentum tensor is derived. All conservation laws follow from the expression under the total divergence. The conservation law depends explicitly on the Lagrangian density which restricts gauge invariant energy-momentum tensors to those which have explicitly gauge invariant Lagrangian densities. In electrodynamics,

$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ is exactly gauge invariant, so this is not a problem.

The equation of motion for the spin-2 model [74] is gauge invariant under the transformation $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. For spin-2, the Fierz-Pauli Lagrangian density ($\mathcal{L} = \frac{1}{4}[\partial_\alpha h^\beta_\beta \partial^\alpha h^\gamma_\gamma - \partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} + 2\partial_\alpha h_{\beta\gamma}\partial^\gamma h^{\beta\alpha} - 2\partial^\alpha h^\beta_\beta \partial^\gamma h_{\gamma\alpha}]$) is not exactly invariant under a gauge transformation [140, 163], even after a change of variables [147]. It is only gauge invariant up to the surface term $\delta_g\mathcal{L} = \partial_\mu[h^{\nu\gamma}\partial_\nu\partial_\gamma\xi^\mu - \frac{1}{2}h^{\mu\nu}\partial_\nu\partial_\gamma\xi^\gamma + \frac{1}{2}h\partial^\mu\partial_\gamma\xi^\gamma - \frac{1}{2}h\Box\xi^\mu]$. This is why the energy-momentum tensor is not gauge invariant [140, 61]. Often this fact is overlooked due to the common priority that only an equation of motion must be found which is gauge invariant. It will be shown in this article why for spin-2 there exists no exactly gauge invariant Lagrangian or gauge invariant conservation law; the canonical energy-momentum tensor depends explicitly on the Lagrangian density [7, 140].

It is clear that exact invariance of the action is a special property, as indicated by electrodynamic theory. The motivation for the current work is as follows: develop a procedure such that an explicitly gauge invariant Lagrangian can be derived. From this procedure, models can be constructed that are invariant under a desired gauge transformation. The power of this procedure is highlighted by a result that was not foreseen by the development of the procedure; the model constructed from a general Lagrangian density which is quadratic in second order derivatives of a symmetric second rank tensor potential (i.e. $\partial_\alpha\partial_\beta h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}$), and invariant under the spin-2 gauge transformation, is uniquely the linearized Gauss-Bonnet gravity model. The Gauss-Bonnet gravity model is a frequent topic in the physics literature [198, 71, 48, 192]; a connection between linearized Gauss-Bonnet gravity and classical electrodynamics can give some additional insight into the significance of the Gauss-Bonnet model.

4.1.2 A Procedure For Gauge Invariant Lagrangian Formulation

To derive a completely gauge invariant model, a procedure was developed that would yield an exactly gauge invariant Lagrangian, equation of motion and energy-momentum tensor. We restrict our attention to Poincaré invariant field theories. The procedure involves defining a linear combination of all possible contractions of terms quadratic in derivatives of fields (i.e. for a model built using a vector potential quadratic in first order derivatives, such as $\partial_\mu A_\nu \partial^\mu A^\nu$). Once the general Lagrangian is constructed, a gauge transformation is applied, and a linear system of equations is obtained. This leads to specific coefficients that will yield a gauge invariant Lagrangian. From here, we obtain a gauge invariant equation of motion, and Noether's theorem is used to derive an energy-momentum tensor [159, 124]. The procedure can be applied for any order N of derivatives/ rank M of tensor potential.

For a model built using a vector potential which is quadratic in first order derivatives ($N =$

1, $M = 1$ i.e. $\partial_\mu A_\nu \partial^\mu A^\nu$), this procedure yields exactly the electrodynamic Lagrangian as the unique gauge invariant combination. For spin-2 ($N = 1$, $M = 2$ i.e. $\partial_\mu h_{\nu\gamma} \partial^\mu h^{\nu\gamma}$), this procedure yields only $\mathcal{L} = 0$. It is possible to modify this procedure to consider Lagrangian densities which are invariant up to some surface term (such as the Fierz-Pauli action), but this does not concern completely gauge invariant models. While spin-2 does not have an explicitly gauge invariant action, a symmetric second rank potential $h_{\nu\gamma}$ does in fact belong to a completely gauge invariant model for $N = 2$, $M = 2$, which requires higher order derivatives in the general Lagrangian ($N = 2$, $M = 2$ i.e. $\partial_\mu \partial_\nu h_{\alpha\beta} \partial^\mu \partial^\nu h^{\alpha\beta}$). The procedure for $N = 2$, $M = 2$ derives a unique completely gauge invariant model, which is exactly the linearized Gauss-Bonnet model for gravity.

Derivation for $N = 1$, $M = 1$ (classical electrodynamics)

For a model with $N = 1$ and $M = 1$ (i.e. vector field theory), the Lagrangian is a scalar which is quadratic in first order field derivatives (i.e. $\partial_\mu A_\nu \partial^\mu A^\nu$). To consider all possible combinations, first write all possible contractions of two indices. From here, write all possible contractions of the next two indices. Since there are only two contracting pairs of indices, this procedure is extremely simple, yielding a linear combination of only 3 possible terms,

$$\mathcal{L} = a\partial_\mu A_\nu \partial^\mu A^\nu + b\partial_\mu A^\mu \partial_\nu A^\nu + c\partial_\mu A_\nu \partial^\nu A^\mu, \tag{4.2}$$

where a, b, c are arbitrary coefficients. Imposing a gauge transformation $A'_\mu = A_\mu + \partial_\mu \phi$ yields a Lagrangian density which can be organized by the original terms ($\mathcal{L}$), and transformation terms ($\delta_g \mathcal{L}$), in the form $\mathcal{L}' = \mathcal{L} + \delta_g \mathcal{L}$,

$$\mathcal{L}' = \mathcal{L} + 2(a + c)\partial_\mu A_\nu \partial^\mu \partial^\nu \phi + 2b\partial_\mu A^\mu \partial_\nu \partial^\nu \phi, \tag{4.3}$$

where $\mathcal{L}$ is given in (4.2). A system of equations is derived such that if all of the $\delta_g \mathcal{L}$ cancel, an exactly gauge invariant Lagrangian density will be obtained. In other words, we must solve for $\delta_g \mathcal{L} = 0$. In order to satisfy this condition, we require $a + c = 0$ and $b = 0$, thus $c = -a$. The resulting Lagrangian density can be factored to $\mathcal{L} = \frac{1}{2} a(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$.

What we find is that the Lagrangian density is exactly the form of classical electrodynamics. The arbitary coefficient a allows for the standard coefficient of the electrodynamic Lagrangian density, $a = -\frac{1}{2}$ yields $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. The choice of the standard coefficient is fixed by equation of motion (Maxwell's equations) and the energy-momentum tensor $T^{\mu\nu} = F^{\mu\alpha} F^\nu_{\ \alpha} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$. The electromagnetic energy-momentum tensor and all other conservation laws related to conformal invariance of Maxwell's equations were first derived from Noether's theorem by Bessel-Hagen [26]. This also can be derived from the canonical Noether energy-momentum tensor after implementing the Belinfante improvement [23]. The procedure therefore can be used to derive a completely gauge invariant model with no free coefficients, in the case of $N = 1$ and $M = 1$, classical electrodynamics.

Derivation for $N = 1$, $M = 2$ (spin-2)

For a model with $N = 1$ and $M = 2$ (i.e. tensor field theory based on symmetric $h_{\nu\gamma}$), the Lagrangian is a scalar which is quadratic in first order field derivatives (i.e. $\partial_\mu h_{\nu\gamma}\partial^\mu h^{\nu\gamma}$). Avoiding redundant contractions yields [97],

$$\mathcal{L} = A\partial_\mu h^\mu_\nu \partial^\nu h^\gamma_\gamma + B\partial_\mu h^\mu_\nu \partial_\gamma h^{\nu\gamma} + C\partial_\mu h^\nu_\nu \partial^\mu h^\gamma_\gamma + D\partial_\mu h_{\nu\gamma}\partial^\mu h^{\nu\gamma} + E\partial_\mu h_{\nu\gamma}\partial^\nu h^{\mu\gamma}, \tag{4.4}$$

where A, B, C, D, E are arbitrary coefficients. Imposing a spin-2 gauge transformation $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ yields $\mathcal{L}' = \mathcal{L} + \delta_g \mathcal{L}$, with $\delta_g \mathcal{L}$ which must vanish in order for a gauge invariant expression to be derived. For clarity, common terms are combined and the D'Alembertian operator ($\Box = \partial_\mu \partial^\mu$) is introduced. The resulting Lagrangian density is,

$$\begin{aligned}\mathcal{L}' = \mathcal{L} &+ A\partial^\nu h \Box \xi_\nu + 2B\partial_\gamma h^{\nu\gamma}\Box\xi_\nu + (A + 4C)\partial_\mu h \partial^\mu \partial_\gamma \xi^\gamma \\ &+ (2A + 2B)\partial_\mu h^{\mu\nu}\partial_\nu\partial_\gamma\xi^\gamma + (4D + 2E)\partial_\mu h_{\nu\gamma}\partial^\mu\partial^\nu\xi^\gamma + 2E\partial_\mu h_{\nu\gamma}\partial^\nu\partial^\gamma\xi^\mu,\end{aligned} \tag{4.5}$$

This equation leads to the homogenous linear system which has either a trivial solution, or a non-trivial solution with free parameter(s). The trivial gauge invariant Lagrangian $\mathcal{L} = 0$, is the only gauge invariant expression. This is the expected result because spin-2 is well known to have an action which is gauge invariant only up to a surface term [61, 140]. We can find this surface term by using integration by parts, leaving us with a total divergence and some remaining terms,

$$\begin{aligned}\mathcal{L}' = \mathcal{L} &+ \partial_\mu[Ah\Box\xi^\mu + 2Bh^{\nu\mu}\Box\xi_\nu + (A + 4C)h\partial^\mu\partial_\gamma\xi^\gamma \\ &+ (2A + 2B)h^{\mu\nu}\partial_\nu\partial_\gamma\xi^\gamma + (4D + 2E)h_{\nu\gamma}\partial^\mu\partial^\nu\xi^\gamma + 2Eh_{\nu\gamma}\partial^\nu\partial^\gamma\xi^\mu] \\ &- (2A + 4C)h\Box\partial_\gamma\xi^\gamma - (2A + 2B + 2E)h^{\mu\nu}\partial_\mu\partial_\nu\partial_\gamma\xi^\gamma - (2B + 4D + 2E)h_{\nu\gamma}\Box\partial^\nu\xi^\gamma.\end{aligned} \tag{4.6}$$

We solve for the coefficients such that the terms not under the total divergence are identically zero. Solving this system of linear equations we have $C = -\frac{1}{2}A$, $D = \frac{1}{2}A$ and $E = -A - B$. For any choice of the free coefficient we have a Lagrangian density which yields the spin-2 equation of motion. If we solve such that the coefficients match up with the conventional coefficient $\frac{1}{2}$ of the spin-2 equation of motion, we must take $A = -\frac{1}{2}$, thus we are left with the solution $A = -\frac{1}{2}$, $B = B$, $C = \frac{1}{4}$, $D = -\frac{1}{4}$ and $E = \frac{1}{2} - B$ with only one free parameter B. We are left with a one parameter family of Lagrangian densities which is invariant up to the surface term,

$$\mathcal{L}' = \mathcal{L} + \partial_\mu[h^{\nu\gamma}\partial_\nu\partial_\gamma\xi^\mu - h^{\mu\nu}\partial_\nu\partial_\gamma\xi^\gamma + \frac{1}{2}h\partial^\mu\partial_\gamma\xi^\gamma - \frac{1}{2}h\Box\xi^\mu \\ + 2B(h^{\mu\nu}\partial_\nu\partial_\gamma\xi^\gamma + h^{\nu\mu}\Box\xi_\nu - h_{\nu\gamma}\partial^\mu\partial^\nu\xi^\gamma - h_{\nu\gamma}\partial^\nu\partial^\gamma\xi^\mu)]. \quad (4.7)$$

The coefficients used to ensure the Lagrangian is invariant up to a surface term from (4.4),

$$\mathcal{L} = \frac{1}{4}[-2\partial_\mu h^\mu_\nu\partial^\nu h^\gamma_\gamma + 4B\partial_\mu h^\mu_\nu\partial_\gamma h^{\nu\gamma} + \partial_\mu h^\nu_\nu\partial^\mu h^\gamma_\gamma - \partial_\mu h_{\nu\gamma}\partial^\mu h^{\nu\gamma} + (2+4B)\partial_\mu h_{\nu\gamma}\partial^\nu h^{\mu\gamma}]. \quad (4.8)$$

We note that for particular value of the free parameter $B = 0$, the Fierz-Pauli Lagrangian is recovered identically. Having a Lagrangian which is explicitly gauge invariant is likely related to the ability to construct a field strength tensor, as will be emphasized during the construction of the gauge invariant model in the following section. Absence of a gauge invariant Lagrangian seems to imply an inability to construct a quadratic combination of independently gauge invariant field strength tensors. The notion of the field strength tensor in a physical model has be alluded to as a physics necessity in the past [58, 131, 195]; the current work can give some more insight into these observations.

Derivation for $N = 2$, $M = 2$ (linearized Gauss-Bonnet)

For a model with $N = 2$ and $M = 2$, the Lagrangian is a scalar which is quadratic in second order field derivatives (i.e. $\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\nu h^{\alpha\beta}$). This can be built from the Lagrangian of the form $\mathcal{L} = M^{\mu\nu\alpha\beta\rho\lambda\sigma\gamma}\partial_\mu\partial_\nu h_{\alpha\beta}\partial_\rho\partial_\lambda h_{\sigma\gamma}$, where $M^{\mu\nu\alpha\beta\rho\lambda\sigma\gamma}$ is all possible permutations of indices of four Minkowski tensors (i.e. $\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\alpha\sigma}\eta^{\beta\gamma}$). Avoiding terms which are redundant after contraction yields,

$$\mathcal{L} = C_1\partial_\mu\partial^\mu h^\nu_\nu\partial_\alpha\partial^\alpha h^\beta_\beta + C_2\partial_\mu\partial^\mu h_{\alpha\beta}\partial_\nu\partial^\nu h^{\alpha\beta} + C_3\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial^\alpha h^\beta_\beta \\ + C_4\partial_\mu\partial_\nu h^\alpha_\alpha\partial_\beta\partial^\beta h^{\mu\nu} + C_5\partial_\mu\partial_\nu h^\nu_\beta\partial_\alpha\partial^\alpha h^{\mu\beta} + C_6\partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial^\nu h^\beta_\beta + C_7\partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial_\beta h^{\nu\beta} \\ + C_8\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + C_9\partial_\mu\partial_\nu h^{\nu\beta}\partial^\mu\partial_\alpha h^\alpha_\beta + C_{10}\partial_\mu\partial_\nu h^\nu_\beta\partial^\beta\partial_\alpha h^{\mu\alpha} \\ + C_{11}\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\nu h^{\alpha\beta} + C_{12}\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\alpha h^{\nu\beta} + C_{13}\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h^{\mu\nu}. \quad (4.9)$$

A spin-2 gauge transformation $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ is then applied. The $\delta_g \mathcal{L}$ consists of 10 unique terms which result in a system of 10 linear equations. These equations decouple into 3 independent systems which we will call i), ii) and iii), each of which are solvable with one free parameter. Not a single coefficient C_n is zero in the case of a nontrivial gauge invariant Lagrangian ($\delta_g \mathcal{L} = 0$). Each of the independent system of linear equations can be solved, i) $C_{12} = -2C_{11},\ C_{13} = C_{11}$, ii) $C_4 = 2C_2,\ C_5 = -4C_2,\ C_6 = C_2,\ C_7 = -4C_2,\ C_9 = 2C_2,\ C_{10} = 2C_2$ and iii) $C_3 = -2C_1,\ C_8 = C_1$. These solutions yield the following independently gauge invariant combinations,

$$\begin{aligned}\mathcal{L} = C_{11}(\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\nu h^{\alpha\beta} - 2\partial_\mu\partial_\nu h_{\alpha\beta}\partial^\mu\partial^\alpha h^{\nu\beta} + \partial_\mu\partial_\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h^{\mu\nu})\\ + C_2(\partial_\mu\partial^\mu h_{\alpha\beta}\partial_\nu\partial^\nu h^{\alpha\beta} + 2\partial_\mu\partial_\nu h^\alpha_\alpha\partial_\beta\partial^\beta h^{\mu\nu} - 4\partial_\mu\partial_\nu h^\nu_\beta\partial_\alpha\partial^\alpha h^{\mu\beta}\\ + \partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial^\nu h^\beta_\beta - 4\partial_\mu\partial_\nu h^\alpha_\alpha\partial^\mu\partial_\beta h^{\nu\beta} + 2\partial_\mu\partial_\nu h^{\nu\beta}\partial^\mu\partial_\alpha h^\alpha_\beta + 2\partial_\mu\partial_\nu h^\nu_\beta\partial^\beta\partial_\alpha h^{\mu\alpha})\\ + C_1(\partial_\mu\partial^\mu h^\nu_\nu\partial_\alpha\partial^\alpha h^\beta_\beta - 2\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial^\alpha h^\beta_\beta + \partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}).\end{aligned} \quad (4.10)$$

The 3 combinations can be factored into contractions of a fourth rank, second rank, and zeroth rank tensor. The motivation is to have each term expressed as the contraction of two field strength tensors. The result of this,

$$\begin{aligned}\mathcal{L} = \frac{1}{4}C_{11}(\partial_\mu\partial_\alpha h_{\nu\beta} + \partial_\nu\partial_\beta h_{\mu\alpha} - \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\nu\partial_\alpha h_{\mu\beta})(\partial^\nu\partial^\beta h^{\mu\alpha} + \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\beta h^{\nu\alpha})\\ + C_2(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h^\alpha_\nu - \partial_\nu\partial_\alpha h^\alpha_\mu)(\Box h^{\mu\nu} + \partial^\mu\partial^\nu h - \partial^\mu\partial^\alpha h^\nu_\alpha - \partial^\nu\partial^\alpha h^\mu_\alpha)\\ + C_1(\Box h - \partial_\mu\partial_\nu h^{\mu\nu})(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}),\end{aligned} \quad (4.11)$$

shows 3 expressions which are familiar to Riemannian geometry. They are the linearized terms of the Riemann tensor $R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha})$, and Ricci tensor $R^{\nu\beta} = \eta_{\mu\alpha}R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\beta\partial^\alpha h^\nu_\alpha + \partial^\nu\partial^\alpha h^\beta_\alpha - \Box h^{\nu\beta} - \partial^\nu\partial^\beta h)$, and Ricci scalar $R = \eta_{\nu\beta}R^{\nu\beta} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h$. Using these linearized Riemann/Ricci tensors, and rewriting the free coefficients as $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ we have,

$$\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2, \quad (4.12)$$

which allows for an infinite number of possible models to be developed. The goal of this work is specific, to find unique combinations which lead to completely gauge invariant models. This condition will now be used to specify possible Lagrangian densities, by use of Noether's theorem. Referring again to (4.20) we have the conservation law for $N = 2, M = 2$,

$$\partial_\omega \left[\eta^{\omega\nu} \mathcal{L} \delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\omega \partial_\lambda h_{\rho\sigma})} \partial_\lambda \delta h_{\rho\sigma} - \left(\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\omega \partial_\lambda h_{\rho\sigma})} \right) \delta h_{\rho\sigma} \right] = 0. \tag{4.13}$$

The first term is gauge invariant because the Lagrangian density is explicitly gauge invariant. The second term can possibly be gauge invariant depending on transformation of the fields $\partial_\lambda \delta h_{\rho\sigma}$ since there will be second order derivatives of $h_{\rho\sigma}$. From (4.12) we calculate,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\omega \partial_\lambda h_{\rho\sigma})} &= 2\tilde{a}[R^{\rho\omega\lambda\sigma} + R^{\lambda\rho\sigma\omega}] \\ &\quad + \tilde{b}[-\eta^{\omega\lambda} R^{\rho\sigma} - \eta^{\rho\sigma} R^{\omega\lambda} + \frac{1}{2}(\eta^{\lambda\sigma} R^{\rho\omega} + \eta^{\lambda\rho} R^{\sigma\omega} + \eta^{\omega\sigma} R^{\rho\lambda} + \eta^{\omega\rho} R^{\sigma\lambda})] \\ &\quad + 2\tilde{c}R[-\eta^{\omega\lambda}\eta^{\rho\sigma} + \frac{1}{2}(\eta^{\omega\rho}\eta^{\lambda\sigma} + \eta^{\omega\sigma}\eta^{\lambda\rho})]. \end{aligned} \tag{4.14}$$

Two identities can be used in the following calculations which are equivalent to the Bianchi identities: $\partial_\omega R^{\lambda\rho\omega\sigma} = \partial^\lambda R^{\rho\sigma} - \partial^\rho R^{\lambda\sigma}$ and $\partial^\rho R = 2\partial_\omega R^{\omega\rho}$. These identities allow for all of the tensors to be expressed as derivatives of the Ricci tensor. Using (4.14), we have for the second term in (4.68),

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\omega \partial_\lambda h_{\rho\sigma})} \partial_\lambda \delta h_{\rho\sigma} &= 2\tilde{a}R^{\omega\rho\lambda\sigma}(\partial_\sigma \delta h_{\lambda\rho} - \partial_\lambda \delta h_{\sigma\rho}) + \tilde{b}R^{\rho\sigma}(\partial_\rho \delta h^\omega_\sigma - \partial^\omega \delta h_{\rho\sigma}) \\ &\quad + \tilde{b}R^{\omega\lambda}(\partial^\rho \delta h_{\rho\lambda} - \partial_\lambda \delta h) + 2\tilde{c}R(\partial_\rho \delta h^{\rho\omega} - \partial^\omega \delta h). \end{aligned} \tag{4.15}$$

If we simply use the canonical transformation $\delta h_{\rho\sigma} = -\partial_\beta h_{\rho\sigma} \delta x^\beta$ we will not have a gauge invariant expression, as in the case of the canonical Noether energy-momentum tensor in electrodynamics. Since we have an explicitly gauge invariant Lagrangian density, the Bessel-Hagen method [26, 110] can be used to derive transformations of the $h_{\rho\sigma}$ which leaves the second term in the conservation law gauge invariant. This procedure involves solving for the field transformations such that coordinate invariance and gauge invariance is simultaneously preserved. The field transformation $\delta h_{\rho\sigma} = -\partial_\beta h_{\rho\sigma} \delta x^\beta + \partial_\rho \xi_\sigma + \partial_\sigma \xi_\rho$ with the most general vector $\xi_\sigma = \tilde{A} h_{\sigma\beta} \delta x^\beta + \tilde{B} h \eta_{\sigma\nu} \delta x^\nu$ must be substituted into (4.15) and solved for the parameters which preserve gauge invariance. We have a unique gauge invariant solution for $\tilde{A} = 1$, $\tilde{B} = 0$. Remarkably the solution is that this transformation is exactly $\delta h_{\rho\sigma} = -2\Gamma^\nu{}_{\rho\sigma} \delta x_\nu$, where $\Gamma^\nu{}_{\rho\sigma} = \frac{1}{2}(\partial^\nu h_{\rho\sigma} - \partial_\rho h^\nu_\sigma - \partial_\sigma h^\nu_\rho)$ is the linearized Christoffel symbol. For the second term in the conservation law we now have,

$$\frac{\partial\mathcal{L}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})}\partial_\lambda\delta h_{\rho\sigma} = (-4\tilde{a}R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} - 2\tilde{b}R_{\rho\sigma}R^{\omega\rho\nu\sigma} - 2\tilde{b}R^{\omega\lambda}R^{\nu}{}_{\lambda} - 4\tilde{c}RR^{\nu\omega})\delta x_\nu. \tag{4.16}$$

The expression in brackets is manifestly gauge invariant without restriction of the free parameters $\tilde{a}, \tilde{b}, \tilde{c}$. We now consider the third term in (4.68) with the transformation $\delta h_{\rho\sigma} = -2\Gamma^{\nu}{}_{\rho\sigma}\delta x_\nu$,

$$\begin{aligned}\left(\partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})}\right)\delta h_{\rho\sigma} = \frac{1}{2}\left(4\tilde{c}+\tilde{b}\right)&\left[2\partial^\omega R\eta^{\rho\sigma} - \partial^\sigma R\eta^{\rho\omega} - \partial^\rho\eta^{\sigma\omega}\right]\Gamma^{\nu}{}_{\rho\sigma}\delta x_\nu \\ &+ \left(4\tilde{a}+\tilde{b}\right)\left[2\partial^\omega R^{\rho\sigma} - \partial^\sigma R^{\rho\omega} - \partial^\rho R^{\sigma\omega}\right]\Gamma^{\nu}{}_{\rho\sigma}\delta x_\nu. \end{aligned} \tag{4.17}$$

The energy-momentum tensor can only be made gauge invariant with respect to the transformation $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ if (4.17) vanishes because second order derivatives of the potential are required for an expression invariant under this transformation [140]. The only way for (4.17) to vanish is therefore to fix the free coefficients of the Lagrangian density. There is a solution such that (4.17) is identically zero (leaving the energy-momentum tensor in lowest integer), $\tilde{a} = \frac{1}{4}, \tilde{b} = -1, \tilde{c} = \frac{1}{4}$,

$$\mathcal{L} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \tag{4.18}$$

which is the linearized form of the Gauss-Bonnet Lagrangian! It was first discovered by Cornelius Lanczos in 1938 [129], and has subsequently been a point of interest in many areas of both physics and mathematics [71, 48, 192, 96, 100, 142]. From Noether's theorem a symmetric, conserved and gauge invariant energy-momentum tensor is derived,

$$T^{\omega\nu} = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\nu\omega} + \frac{1}{4}\eta^{\omega\nu}(R_{\mu\lambda\alpha\beta}R^{\mu\lambda\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \tag{4.19}$$

The Belinfante procedure for higher order gravity can also be used to obtain this energy-momentum tensor [23, 7], as well as the Fock method for deriving an energy-momentum tensor which is symmetric and conserved on shell [75, 28]. From both methods the resulting energy-momentum tensor is exactly what we have in (4.27).

Just like in classical electrodynamics the procedure for $N = 2, M = 2$ allows for the derivation of a completely gauge invariant model, where the energy-momentum tensor is symmetric, conserved and gauge invariant. The energy-momentum tensor presented in (4.27) is a well known expression to string theorists for several decades [172, 34, 156]. The fact that it follows from a procedure originally developed for deriving completely gauge invariant models in relativistic field theories (such as electrodynamics) was a completely unforeseen result. The connection between electrodynamics and Gauss-Bonnet gravity models, as well as the meaning of this connection, is worth further investigation.

4.1.3 Conclusions

A procedure was developed for building completely gauge invariant models by imposing gauge invariance and Noether's theorem to general scalar Lagrangian densities. The electrodynamic Lagrangian density is shown to that follows from this procedure for $N = 1$ and $M = 1$, which leads directly to the completely gauge invariant theory. For spin-2 ($N = 1$, $M = 2$), this procedure yields no nontrivial results, which is expected because the spin-2 Lagrangian density is only invariant up to a surface term. A model with $M = 2$ rank of potential (second rank symmetric potential $h_{\mu\nu}$) and $N = 2$ order of derivatives was derived from the procedure, yielding 3 possible contractions: linearized Riemann tensors, Ricci tensors, and Ricci scalars. It is found that for a specific combination of these terms, a completely gauge invariant model can be constructed analogous to electrodynamics; a model which is exactly the linearized Gauss-Bonnet gravity model.

If a Lagrangian density is built from a tensor potential of rank M and order of derivatives $N = M$, the procedure can be used to derive a completely gauge invariant model; the Lagrangian density, equation of motion and energy-momentum tensor are all gauge invariant. This was highlighted by the fact that for $N = M = 1$ the procedure yields classical electrodynamics, and for $N = M = 2$ with a totally symmetric tensor potential the procedure yields linearized Gauss-Bonnet gravity. We note that this pattern continues for totally symmetric tensor potentials when $N = M > 2$ and is the subject of future work. In cases where $N \neq M$ it is possible to derive completely gauge invariant models if $N > M$, or for totally antisymmetric tensor potentials of any rank M when $N = 1$ (keeping in mind the rank M of totally antisymmetric potentials is restricted by the dimension D of a theory as $M < D$).

A major characteristic highlighted by the developed procedure is the importance of a Lagrangian that is exactly gauge invariant in physical field theories; not simply invariant up to some surface term. This is explicit, but rarely discussed, in the from of the canonical energy-momentum tensor. The first term in (4.68) must be independently gauge invariant, otherwise the only possible gauge invariant energy momentum tensor will be $T^{\lambda\gamma} = 0$. Electrodynamics ($N = 1$, $M = 1$) is a completely gauge invariant field theory built from a unique gauge invariant Lagrangian density. The gauge invariant Lagrangian for the $N = 2$ and $M = 2$ model presented in this article not only has this attribute, but it is directly related to the building blocks of general relativity through linearized Riemann tensors that were derived. Uniqueness found in the gauge invariant Lagrangian density expressions suggests that it is not enough to consider only the gauge invariance of the equation of motion. Complete gauge invariance of these models emphasizes the need for both exactly gauge invariant Lagrangians and conservation laws. Gauge invariant Lagrangian densities produced by this procedure imply existence of gauge invariant field strength tensors that can be used to build the corresponding model. The connection between Gauss-Bonnet gravity and classical electrodynamics is the primary point of interest presented in this article and will be the subject of future work.

4.2 A connection between linearized Gauss-Bonnet gravity and classical electrodynamics II: Complete dual formulation

Abstract In a recent publication a procedure was developed which can be used to derive completely gauge invariant models from general Lagrangian densities with N order of derivatives and M rank of tensor potential. This procedure was then used to show that unique models follow for each order, namely classical electrodynamics for $N = M = 1$ and linearized Gauss-Bonnet gravity for $N = M = 2$. In this article, the nature of the connection between these two well explored physical models is further investigated by means of an additional common property; a complete dual formulation. First we give a review of Gauss-Bonnet gravity and the dual formulation of classical electrodynamics. The dual formulation of linearized Gauss-Bonnet gravity is then developed. It is shown that the dual formulation of linearized Gauss-Bonnet gravity is analogous to the homogenous half of Maxwell's theory; both have equations of motion corresponding to the (second) Bianchi identity, built from the dual form of their respective field strength tensors. In order to have a dually symmetric counterpart analogous to the non-homogenous half of Maxwell's theory, the first invariant derived from the procedure in $N = M = 2$ can be introduced. The complete gauge invariance of a model with respect to Noether's first theorem, and not just the equation of motion, is a necessary condition for this dual formulation. We show that this result can be generalized to the higher spin gauge theories, where the spin-n curvature tensors for all $N = M = n$ are the field strength tensors for each n. These completely gauge invariant models correspond to the Maxwell-like higher spin gauge theories whose equations of motion have been well explored in the literature.

4.2.1 Motivation

In [14], a procedure for deriving completely gauge invariant models from general linear combinations of derivatives of order N and rank of potential M was developed. Complete gauge invariance occurs for a model when the Lagrangian density, equation of motion and energy-momentum tensor are all independently and exactly gauge invariant. The procedure involves solving for the free coefficients in the linear combination with respect to Noether's (first) theorem [159, 124] such that the model is completely gauge invariant under a particular gauge transformation. In the case of $N = M = 1$ under a spin-1 gauge transformation, electrodynamics is uniquely derived from the procedure. In the case of $N = M = 2$, under a spin-2 gauge transformation (sometimes referred to as linearized diffeomorphisms), linearized Gauss-Bonnet gravity is uniquely derived.

The connection of these models to a common procedure raised an obvious question, what is the reason for this connection, and why Gauss-Bonnet gravity of the many metric theories of gravity that exist in the literature. The present article attempts to answer both questions through the Gauss-Bonnet theorem, and the additional non-trivial property shared by these two models, complete dual formulation of the Lagrangian, equation of motion and energy-momentum tensor. Once again these models will be derived from the Noether identity from Noether's first theorem, given below for a general potential Φ_A [14],

$$\left(\frac{\partial \mathcal{L}}{\partial \Phi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} + \partial_\mu \partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \Phi_A)} + \dots \right) \delta\Phi_A$$
$$+ \partial_\mu \left(\eta^{\mu\nu} \mathcal{L} \delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \Phi_A)} \partial_\omega \delta\Phi_A - \left[\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\omega \Phi_A)} \right] \delta\Phi_A + \dots \right) = 0 \,. \quad (4.20)$$

The article will be structured as follows. In Section 4.2.2 an overview of Gauss-Bonnet gravity is given, with its connection to the Gauss-Bonnet theorem, and how it is derived from the Euler class $e(\Omega)$ in the integrand of the Gauss-Bonnet theorem. Next the dual linearized Riemann tensor is introduced and connected to the original results of Lanczos that first noted these connections between the dual Riemann tensors, their scalars, and what is now known as the Gauss-Bonnet Lagrangian. In Section 4.2.3 an overview of the dual formulation of electrodynamics is presented, and how every component of the theory with respect to Noether's theorem (the Lagrangian density, equation of motion and energy-momentum tensor) can be expressed explicitly in dual form.

Section 4.2.4 is dedicated to converting the linearized Gauss-Bonnet gravity model from [14] into dual form with respect to Noether's first theorem. It is shown that this model has the same general dual formulation as the homogenous half of Maxwell's theory; the equation of motion is the second Bianchi identity built from the dual linearized Riemann tensor tensor. In Section 4.2.5 possible invariants derived from the procedure in [14] are discussed that can give complete dual formulation analogous to the complete dual formulation of electrodynamics. Indeed the invariant $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ yields this with respect to the Lagrangian and equation of motion, but complications arise with the third term in the energy-momentum tensor from Noether's first theorem. This is because, as shown in [14], only for very particular Lagrangian densities can this be made gauge invariant and symmetric, namely the Gauss-Bonnet combination. Two possible remedies to this problem are given and possible ramifications are discussed.

In Section 4.2.6 the internal dual formulations of the respective electodynamics and linearized gravity models are presented. Section 4.2.7 gives the general forms for the analogous expressions between the two models which are generalized for any independently gauge invariant spin-n field strength tensor. These models are the Maxwell-like higher spin gauge theories for the spin-n curvature tensors [82, 79, 21]. The complete dual and gauge invariance of these models with respect to Noether's theorem provides more compelling evidence for the requirement of complete invariance properties of physical theories with respect to all components: the Lagrangian density, equation of motion and energy-momentum tensor of the model.

4.2.2 Gauss-Bonnet gravity and the dual Riemann tensors

We begin by providing details regarding the origin of Gauss-Bonnet gravity, its relation to the Gauss-Bonnet theorem, and how the common Lagrangian in the literature is obtained from the Euler class in the integrand of the theorem. This is necessary because the required calculations for our article are scattered throughout the literature, if at all. The book by Eguchi, Gilkey and Hanson [67] will be taken as the primary reference for details here, however even this reference is missing considerable detail and explanation. The Gauss-Bonnet theorem got its name by the work of Gauss in 1827 (Gauss's theorem egregrium) [86] and Bonnet in 1848 [31], although neither of these presentations are what we refer to as the Gauss-Bonnet theorem in the present day (they were earlier developments of the theorem). The modern day version was first presented by Dyck in 1890 [66] for the specific case of R^3, and finally to n dimensions by Hopf in 1926 [106], with the proof of the general formula for Riemannian manifolds being completed by Chern [49].

It was Allendoerfer [4] who first showed that the integrand for a Riemannian manifold of dimension d is the general expression of which the special case $d = 4$ is what we will derive below (the Gauss-Bonnet Lagrangian). This was recognized for $d = 4$ indirectly by Lanczos [129] a couple years earlier, but from motivations discussed later in this section. A more detailed account of this history was given by [194]. The modern form of the Gauss-Bonnet theorem is sometimes referred to as the generalized Gauss-Bonnet theorem or Chern-Gauss-Bonnet theorem to make a distinction between the more advanced (modern) version compared to the work of Gauss and Bonnet. The modern form of the theorem states,

$$\chi(M) = \int_{\bar{M}} e(\Omega), \tag{4.21}$$

where χ is the Euler characteristic of manifold $\bar{M}$ and $e(\Omega)$ is the Euler class. The Euler class can be expressed in terms of the Pfaffian of the curvature form $Pf(\Omega)$,

$$e(\Omega) = \frac{1}{(2\pi)^{d/2}} Pf(\Omega). \tag{4.22}$$

The Pfaffian of the curvature form for a Riemannian manifold in 4 dimensions (4D) is the

$d = 4$ case. The expression for the Euler class of the curvature form in this case was first given by Allendoerfer [4], but in more explicit notation by Eguchi, Gilkey and Hanson [67]. The Pfaffian is given by,

$$Pf(\Omega) = \frac{1}{8}\epsilon_{abcd}\Omega^{ab} \wedge \Omega^{cd}, \tag{4.23}$$

where the curvature 2-form for the Riemannian manifold is given in terms of the Riemann tensor, $\Omega^{\mu\nu} = \frac{1}{2}R^{\mu\nu}{}_{\rho\sigma}dx^{\rho} \wedge dx^{\sigma}$. The Euler class therefore reads,

$$e(\Omega) = \frac{1}{4}\frac{1}{32\pi^2}\epsilon_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\alpha\beta}{}_{\lambda\gamma}dx^{\rho} \wedge dx^{\sigma} \wedge dx^{\lambda} \wedge dx^{\gamma}. \tag{4.24}$$

Expanding out this summation yields,

$$\begin{aligned} e(\Omega) = \frac{1}{32\pi^2}[8(&R^{12}{}_{34}R^{34}{}_{12} + R^{14}{}_{23}R^{23}{}_{14} + R^{13}{}_{24}R^{24}{}_{13} - R^{12}{}_{13}R^{34}{}_{24} + R^{12}{}_{14}R^{34}{}_{23} + R^{12}{}_{23}R^{34}{}_{14} \\ &- R^{12}{}_{24}R^{34}{}_{13} + R^{13}{}_{12}R^{42}{}_{34} + R^{13}{}_{14}R^{42}{}_{23} + R^{13}{}_{23}R^{42}{}_{14} + R^{13}{}_{34}R^{42}{}_{12} + R^{14}{}_{12}R^{23}{}_{34} - R^{14}{}_{13}R^{23}{}_{24} \\ &- R^{14}{}_{24}R^{23}{}_{13} + R^{14}{}_{34}R^{23}{}_{12} + R^{12}{}_{12}R^{34}{}_{34} + R^{13}{}_{13}R^{24}{}_{24} + R^{14}{}_{14}R^{23}{}_{23})]dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \end{aligned} \tag{4.25}$$

What is in square brackets above is identically $[8(\dots)] = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. Additionally, using the relationship $d^4x = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = \frac{1}{24}\epsilon_{\mu\nu\alpha\beta}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\alpha} \wedge dx^{\beta}$, the Euler class for the Riemannian manifold in 4D is,

$$e(\Omega) = \frac{1}{32\pi^2}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)d^4x. \tag{4.26}$$

Therefore the Gauss-Bonnet theorem for this case reads $\chi(M) = \frac{1}{32\pi^2}\int_{\tilde{M}}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)d^4x$. It is this integrand of the Gauss-Bonnet theorem in 4D that is precisely the Lagrangian density for the Gauss-Bonnet gravity model.

This contribution was first introduced to the physics community by Cornelius Lanczos in 1938 [129], although happened across by very different means. Lanczos was considering various invariants that can be obtained from the Riemannian tensors, as presented in his paper $I_1 = R_{\mu\nu}R^{\mu\nu}$, $I_2 = R^2$ and $I_3 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. Of course these are the 3 invariants found above in the Gauss-Bonnet theorem, and the 3 invariants derived in [14] as $\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$, where $\tilde{a} = \frac{1}{4}$, $\tilde{b} = -1$ and $\tilde{c} = \frac{1}{4}$ for linearized Gauss-Bonnet gravity (Equation (4.43)).

What Lanczos noticed is that if we consider a Lagrangian density formed from the combination $I_3 - 4I_1 + I_2$, it will make no contribution to the equation of motion. This result is now more appropriately understood as the nature of topological invariants, which can be expressed as a total derivative in the action. This result, however, as emphasized in [14], does not mean that the action will not contribute to the energy-momentum tensor of the model. In addition, as we show in Section 4.2.4, the equation of motion (while zero) is in fact the second Bianchi identity analogous to the homogenous half of Maxwell's equations. The precise form of the energy-momentum tensor is a well known expression to string theorists for several decades [172, 198, 34, 156], as derived for linearized Gauss-Bonnet gravity from Noether's first theorem in [14],

$$T^{\omega\nu} = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu} + \frac{1}{4}\eta^{\omega\nu}(R_{\mu\lambda\alpha\beta}R^{\mu\lambda\alpha\beta} - 4R_{\mu\gamma}R^{\mu\gamma} + R^2). \quad (4.27)$$

Gauss-Bonnet gravity is an extensively published model in the literature, interests which have only been increasing in recent years [48, 50, 96, 142, 24, 91]. This past year [91] has attracted significant attention in the literature by claiming the the Gauss-Bonnet model can be used to predict 'new' gravitational dynamics solving which can explain several still unexplained phenomena. We note that many authors have been writing to support, criticize and further this result [73].

Lanczos did consider two additional invariants built from dual tensors [129],

$$\mathbf{R}_{\alpha\beta\mu\nu} = \frac{1}{2}R^{\rho\sigma}{}_{\mu\nu}\epsilon_{\rho\sigma\alpha\beta}, \quad (4.28)$$

$$\mathcal{R}_{\mu\nu\alpha\beta} = \frac{1}{4}R^{\rho\sigma\lambda\gamma}\epsilon_{\rho\sigma\mu\nu}\epsilon_{\alpha\beta\lambda\gamma}, \quad (4.29)$$

which he called 'simply' dual $\mathbf{R}^{\alpha\beta\mu\nu}$ and 'doubly' dual $\mathcal{R}^{\alpha\beta\mu\nu}$, respectively. The two invariants he considered were each of these contracted with the Riemann tensor, $K_1 = \mathbf{R}^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ and $K_2 = \mathcal{R}^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$. From this he showed that the invariant K_2 can be used to express the combination which makes no contribution to the equation of motion $K_2 = I_3 - 4I_1 + I_2$. This will be the starting point for Section 4.2.4 where the linearized Gauss-Bonnet model will be completely rewritten into an explicit dual formulation, as in the case of dual electrodynamics. In order to do this, Section 4.2.3 will first present the complete electrodynamics model in dual form.

4.2.3 Dual electrodynamics

Dual electrodynamic scalars

The dual formulation of electrodynamics has an interesting history. Heaviside first noticed the dual invariance of the complete 8 Maxwell equations when he first wrote them in vector form [102]. Maxwell's equations were presented from the both the field strength tensor $F^{\mu\nu}$ (the 4 non-homogenous equations in Equation (4.36)) and dual tensor $\mathcal{F}^{\mu\nu}$ (the 4 homogenous equations in Equation (4.40)) by Minkowski in [150]. Later the complete 8 Maxwell equations were reformulated into a single field strength tensor $F^{\mu\nu}$ by Einstein [69]. The appeal here was that a single field strength tensor $F^{\mu\nu}$ could be defined from which all of Maxwell's equations could be presented. The downside was that explicit dual invariance of the model was hidden as a consequence, and that the homogenous half of Maxwell's equations were presented simply as a property of the field strength tensor (from the second Bianchi identity $\partial_\sigma F_{\alpha\beta} + \partial_\beta F_{\sigma\alpha} + \partial_\alpha F_{\beta\sigma} = 0$), rather than following in the Euler-Lagrange equation from a fundamental Lagrangian density.

Since this time, the Lagrangian density considered to be fundamental to electrodynamics is $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ as derived from the procedure in [1], where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. This Lagrangian density yields the non-homogenous half of Maxwell's equations in Equation (4.36) from the Euler-Lagrange equation. Considering the dual tensor of electrodynamics $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, it is possible build in principle 3 invariants, $M_1 = F_{\alpha\beta}F^{\alpha\beta}$, $M_2 = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ and $M_3 = F_{\mu\nu}\mathcal{F}^{\mu\nu}$. It is well known that the first two can be expressed in terms of one another as $M_1 = -M_2$. The common objection to M_3 is that it should not be included in the action because it can change sign under an odd numbered parity transformation, since it is formed from the inner product of polar vector $\vec{E}$ and axial vector $\vec{B}$. This sign change however, does not effect the equation of motion. Since Lagrangians which are not exactly gauge invariant but admit gauge invariant equations of motion (such as the spin-2 Fierz-Pauli action invariant up to a surface term [14, 163, 140, 61]) are well accepted in the literature, a change of sign is also a negligible problem if it does not affect the physical model. We will not focus on this philosophical question in this article.

Note that the authors of a highly cited paper on dual electrodynamics [46, 30] propose a new Lagrangian of the form $M_1 + M_2$ where the dual field strength is redefined as $\mathcal{F}_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ in terms of a second 4-potential C_μ. This has been proposed by numerous other authors throughout the years without gaining much traction. From this perspective the basic idea is that variation with respect to both A_μ and C_μ of the Lagrangian $M_1 + M_2$ will yield all 8 of Maxwell's equations, the 4 non-homogenous equations from variation with respect to A_μ and the 4 homogenous equations from variation with respect to C_μ. This differs from the procedure in [14] and that general view that electrodynamics is built from a single potential A_μ, so the presentation in [46, 30] will not be considered here. If the general Lagrangian density in [14] is built using both potentials in separate terms, with the gauge transformation $C'_\mu = C_\mu + \partial_\mu\phi$, then their presentation can also be derived. Their thesis, however, that electrodynamics should be conventionally expressed in dual invariant form to (i) obtain all of Maxwell's equation from the variational approach (since $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ only yields the non-homogenous equations), and (ii) allow for a more symmetric presentation of all conservation laws, is hard to argue. The following presentation of the dual formulation is perhaps superior given the complete derivation of Maxwell's theory from M_1, M_2, M_3 without the need to introduce any non-canonical potential vectors [89, 84, 126].

Generalized Kronecker delta

In order to perform many of the calculations involving 4D dual expressions for classical electrodynamics and the linearized gravity models discussed in this article, it is necessary to review the generalized Kronecker delta in 4D for a Minkowski spacetime. The generalized Kronecker delta is defined as the determinant of the Kronecker deltas of the permuted indices as follows,

$$\delta^{\alpha\beta\lambda\gamma}_{\rho\sigma\mu\nu} = \left\| \begin{pmatrix} \delta^\alpha_\rho & \delta^\beta_\rho & \delta^\lambda_\rho & \delta^\gamma_\rho \\ \delta^\alpha_\sigma & \delta^\beta_\sigma & \delta^\lambda_\sigma & \delta^\gamma_\sigma \\ \delta^\alpha_\mu & \delta^\beta_\mu & \delta^\lambda_\mu & \delta^\gamma_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\lambda_\nu & \delta^\gamma_\nu \end{pmatrix} \right\| . \tag{4.30}$$

The product of two Levi-Civita symbols is defined in terms of the generalized Kronecker delta, however in the case of Minkowski spacetime this relationship has a sign change, since raising the indices in one of the symbols will produce an overall sign change. Therefore for 4D Minkowski spacetime follows the relationship $\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\lambda\gamma} = -\delta^{\alpha\beta\lambda\gamma}_{\rho\sigma\mu\nu}$. Computing the determinant above is straightforward and yields,

$$\begin{aligned}
\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\lambda\gamma} = -\delta^{\alpha\beta\lambda\gamma}_{\rho\sigma\mu\nu} = \\
&- \delta^\alpha_\rho\delta^\beta_\sigma\delta^\lambda_\mu\delta^\gamma_\nu + \delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\mu\delta^\lambda_\nu - \delta^\alpha_\rho\delta^\gamma_\sigma\delta^\beta_\mu\delta^\lambda_\nu + \delta^\alpha_\rho\delta^\gamma_\sigma\delta^\lambda_\mu\delta^\beta_\nu - \delta^\alpha_\rho\delta^\lambda_\sigma\delta^\gamma_\mu\delta^\beta_\nu + \delta^\alpha_\rho\delta^\lambda_\sigma\delta^\beta_\mu\delta^\gamma_\nu \\
&- \delta^\lambda_\rho\delta^\alpha_\sigma\delta^\beta_\mu\delta^\gamma_\nu + \delta^\lambda_\rho\delta^\alpha_\sigma\delta^\gamma_\mu\delta^\beta_\nu - \delta^\lambda_\rho\delta^\gamma_\sigma\delta^\alpha_\mu\delta^\beta_\nu + \delta^\lambda_\rho\delta^\gamma_\sigma\delta^\beta_\mu\delta^\alpha_\nu - \delta^\lambda_\rho\delta^\beta_\sigma\delta^\gamma_\mu\delta^\alpha_\nu + \delta^\lambda_\rho\delta^\beta_\sigma\delta^\alpha_\mu\delta^\gamma_\nu \\
&- \delta^\beta_\rho\delta^\lambda_\sigma\delta^\alpha_\mu\delta^\gamma_\nu + \delta^\beta_\rho\delta^\lambda_\sigma\delta^\gamma_\mu\delta^\alpha_\nu - \delta^\beta_\rho\delta^\gamma_\sigma\delta^\lambda_\mu\delta^\alpha_\nu + \delta^\beta_\rho\delta^\gamma_\sigma\delta^\alpha_\mu\delta^\lambda_\nu - \delta^\beta_\rho\delta^\alpha_\sigma\delta^\gamma_\mu\delta^\lambda_\nu + \delta^\beta_\rho\delta^\alpha_\sigma\delta^\lambda_\mu\delta^\gamma_\nu \\
&\quad - \delta^\gamma_\rho\delta^\beta_\sigma\delta^\alpha_\mu\delta^\lambda_\nu + \delta^\gamma_\rho\delta^\beta_\sigma\delta^\lambda_\mu\delta^\alpha_\nu - \delta^\gamma_\rho\delta^\lambda_\sigma\delta^\beta_\mu\delta^\alpha_\nu + \delta^\gamma_\rho\delta^\lambda_\sigma\delta^\alpha_\mu\delta^\beta_\nu - \delta^\gamma_\rho\delta^\alpha_\sigma\delta^\lambda_\mu\delta^\beta_\nu + \delta^\gamma_\rho\delta^\alpha_\sigma\delta^\beta_\mu\delta^\lambda_\nu .
\end{aligned} \tag{4.31}$$

If two of these indices are contracted the above expression simplifies to,

$$\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\lambda\nu} = -\delta^{\alpha\beta\lambda\nu}_{\rho\sigma\mu\nu} = \delta^\lambda_\rho\delta^\beta_\sigma\delta^\alpha_\mu - \delta^\beta_\rho\delta^\lambda_\sigma\delta^\alpha_\mu - \delta^\lambda_\rho\delta^\alpha_\sigma\delta^\beta_\mu + \delta^\alpha_\rho\delta^\lambda_\sigma\delta^\beta_\mu + \delta^\beta_\rho\delta^\alpha_\sigma\delta^\lambda_\mu - \delta^\alpha_\rho\delta^\beta_\sigma\delta^\lambda_\mu . \tag{4.32}$$

If an additional two indices are contracted we are left with,

$$\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\mu\nu} = -\delta^{\alpha\beta\mu\nu}_{\rho\sigma\mu\nu} = -2(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}). \tag{4.33}$$

These expressions will be used to form identities between the dual and non-dual expressions. For example, the aforementioned relationship $M_1 = F_{\mu\nu}F^{\mu\nu} = -M_2 = -\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ can be readily computed with $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ as,

$$-\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{1}{4}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F^{\rho\sigma} = \frac{1}{2}(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma})F_{\alpha\beta}F^{\rho\sigma} = F_{\mu\nu}F^{\mu\nu}. \tag{4.34}$$

These generalized Kronecker deltas will be referred to throughout the article.

Dualizing the non-homogenous half of electrodynamics

The conventional Lagrangian for electrodynamic theory $\mathcal{L}_{MNH} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ yields only half of Maxwell's equations in the Euler-Lagrange equation, namely the 4 non-homogenous equations known as the Gauss-Ampere laws in Equation (4.36). Note that reference to anything associated to the non-homogenous half of Maxwell's equations from here forward will be denoted with subscript MNH = Maxwell's non-homogenous for clarity. This Lagrangian is, of course, perfectly sound at deriving half of the theory from Noether's theorem, as shown in [14]. This non-homogenous equation of motion is dual to the equation of motion that represents the 4 homogenous equations, derived in Section 4.2.3.

Recall from Equation (4.34) the relationship $F_{\mu\nu}F^{\mu\nu} = -\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$. Since $\mathcal{L}_{MNH} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ doesn't need to be changed, it can be expressed equivalently as $\mathcal{L}_{MNH} = \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ or $\mathcal{L}_{MNH} = \frac{1}{8}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - F_{\mu\nu}F^{\mu\nu})$. All of these options change sign under $F_{\mu\nu} \leftrightarrow \mathcal{F}_{\mu\nu}$, however the third is preferred since it is explicitly in dual form. Therefore the dual Lagrangian density is defined as the third option,

$$\mathcal{L}_{MNH} = \frac{1}{8}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - F_{\mu\nu}F^{\mu\nu}). \tag{4.35}$$

The equations of motion that follows from substitution of this Lagrangian density into the Euler-Lagrange equation in Equation (4.20) are the 4 equations known as the non-homogenous half of Maxwell's equations (Gauss-Ampere laws). These equations are sourced by the 4-current J^{μ} that is coupled in to the conventional Lagrangian $\mathcal{L}_{MNH}$ via $A_{\mu}J^{\mu}$, hence the name non-homogenous. As in [14] we are only concerned with the free fields derived from the procedure, thus we will focus on the equations of motion E^{ρ}_{MNH} that follows from $\mathcal{L}_{MNH}$ in

Equation (4.35) substituted into the Euler-Lagrange equation (Equation (4.20)),

$$E^{\rho}_{MNH} = \partial_{\sigma} F^{\sigma\rho}. \tag{4.36}$$

Dual formulation of Maxwell's equations require the complete 8 equations, therefore the homogenous 4 are required, which are presented in Section 4.2.3. Finally, consider the energy-momentum tensor $T^{\mu\nu}_{MNH} = F^{\mu\alpha}F^{\nu}_{\ \alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ derived from Noether's first theorem using Equations (4.20) and (4.35). This can be elegantly dualized by deriving an identity relating the two terms above to the dual expression $\mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}_{\alpha}$ via equation (4.33) ,

$$\mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}_{\ \alpha} = \frac{1}{4}\eta^{\nu\omega}\epsilon^{\rho\beta\mu\alpha}\epsilon_{\xi\sigma\omega\alpha}F_{\rho\beta}F^{\xi\sigma} = \frac{1}{4}\eta^{\nu\omega}(-\delta^{\rho\beta\mu\alpha}_{\xi\sigma\omega\alpha})F_{\rho\beta}F^{\xi\sigma} = -\frac{1}{2}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu\alpha}F^{\nu}_{\ \alpha}. \tag{4.37}$$

From this expression the term proportional to Minkowski can be re-expressed as $-\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}\mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}_{\ \alpha} - \frac{1}{2}F^{\mu\alpha}F^{\nu}_{\ \alpha}$. Therefore the result for the dualized energy-momentum tensor is,

$$T^{\mu\nu}_{MNH} = \frac{1}{2}[F^{\mu\alpha}F^{\nu}_{\ \alpha} + \mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}_{\ \alpha}]. \tag{4.38}$$

This equation is the dual form of the conventional energy-momentum tensor in electrodynamic theory and it is symmetric, conserved, gauge invariant, and has the additional explicit property of dual invariance under interchange $F_{\mu\nu} \leftrightarrow \mathcal{F}_{\mu\nu}$.

Dualizing the homogenous half of electrodynamics

In order to have the dual symmetry with the non-homogenous half of Maxwell's equations, the dual equation of motion is required, namely the 4 homogenous equations known as the Gauss-Faraday laws (Equation (4.40)) [150]. The only remaining invariant, $M_3 = F_{\mu\nu}\mathcal{F}^{\mu\nu}$, gives precisely this equation of motion in the Euler-Lagrange equation. This completes the dual symmetry of the equations of motion. Reference to anything associated to the homogenous half of Maxwell's equations from here forward will be denoted with subscript MH = Maxwell's homogenous for clarity. The current section presents the dual form of the Lagrangian density $\mathcal{L}_{MH}$, equation of motion E^{ρ}_{MH}, and energy-momentum tensor $T^{\mu\nu}_{MH}$. Starting with the Lagrangian for the homogenous half of Maxwell's equations,

$$\mathcal{L}_{MH} = -\frac{1}{4}F_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{4.39}$$

Inserting this expression into the Euler-Lagrange equation in Equation (4.20), $\frac{\partial\mathcal{L}_{MH}}{\partial A_{\rho}} - \partial_{\sigma}\frac{\partial\mathcal{L}_{MH}}{\partial(\partial_{\sigma}A_{\rho})} = \partial_{\sigma}\mathcal{F}^{\sigma\rho}$. Therefore Maxwell's homogenous equations are indeed the dual to the non-homogenous,

$$E^{\rho}_{MH} = \partial_{\sigma}\mathcal{F}^{\sigma\rho}. \tag{4.40}$$

Note that using the definiton of the dual field strength tensor $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, this equation can be re-expressed in terms of the Bianchi identity via $E^{\rho}_{MH} = \frac{1}{2}\epsilon^{\alpha\beta\sigma\rho}\partial_{\sigma}F_{\alpha\beta} = \frac{1}{6}\epsilon^{\alpha\beta\sigma\rho}(\partial_{\sigma}F_{\alpha\beta} + \partial_{\beta}F_{\sigma\alpha} + \partial_{\alpha}F_{\beta\sigma})$. Therefore an alternate form of the homogenous equations are in terms of the Bianchi identity as follows,

$$E^{\rho}_{MH} = \frac{1}{6}\epsilon^{\alpha\beta\sigma\rho}(\partial_{\sigma}F_{\alpha\beta} + \partial_{\beta}F_{\sigma\alpha} + \partial_{\alpha}F_{\beta\sigma}) = 0. \tag{4.41}$$

Commonly in the literature the homogenous equations are expressed as $\partial_{\sigma}F_{\alpha\beta} + \partial_{\beta}F_{\sigma\alpha} + \partial_{\alpha}F_{\beta\sigma} = 0$, due to the Bianchi identity representing the homogenous half of Maxwell's equations, this was the aforementioned idea of Einstein [69]. The problem with this approach is that it cares not if half of Maxwell's theory is derived by the Euler-Lagrange equation; instead half of the theory is simply stated separately as a property of the field strength tensor. The dual formulation solves this problem elegantly.

Finally, an energy-momentum tensor can be derived from Noether's first theorem using Equations (4.20) and (4.39),

$$T^{\mu\nu}_{MH} = \mathcal{F}^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}\mathcal{F}^{\alpha\beta}F_{\alpha\beta}, \tag{4.42}$$

which is also dually invariant under interchange $F_{\mu\nu} \leftrightarrow \mathcal{F}_{\mu\nu}$.

4.2.4 Dual linearized Gauss-Bonnet gravity

Dualizing the Lagrangian density

Now that the complete theory of electrodynamics has been expressed explicitly in the dual formulation, the completely gauge invariant linearized Gauss-Bonnet gravity model derived in [1] can be dualized. Recall that the Lagrangian was of the form

$$\mathcal{L}_{LGB} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \tag{4.43}$$

where reference to anything associated to the linearized Gauss-Bonnet gravity model from here forward will have subscript LGB = linearized Gauss-Bonnet for clarity. The scalars here are built from the linearized Riemann tensor $R^{\mu\nu\alpha\beta}$, linearized Ricci tensor $R^{\nu\beta}$, and linearized

Ricci scalar R, respectively,

$$R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha}), \tag{4.44}$$

$$R^{\nu\beta} = \eta_{\mu\alpha}R^{\mu\nu\alpha\beta} = \frac{1}{2}(\partial^\beta\partial^\alpha h^\nu_\alpha + \partial^\nu\partial^\alpha h^\beta_\alpha - \Box h^{\nu\beta} - \partial^\nu\partial^\beta h), \tag{4.45}$$

$$R = \eta_{\nu\beta}R^{\nu\beta} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \tag{4.46}$$

In the current section we will derive the dual form of the Lagrangian density $\mathcal{L}_{LGB}$, equation of motion $E^{\rho\sigma}_{LGB}$, and energy-momentum tensor $T^{\mu\nu}_{LGB}$. In order to dualize this Lagrangian, identities can be derived for the 'doubly' dual linearized Riemann tensor $\mathcal{R}_{\mu\nu\alpha\beta}$, as well as the corresponding Ricci tensors and Ricci scalars. For brevity the 'doubly' dual Riemann tensor $\mathcal{R}_{\mu\nu\alpha\beta}$ in Equation (4.29) will be referred to as the dual Riemann tensor; this is the dual tensor which we use in our article. From here the dual Ricci tensor $\mathcal{R}_{\mu\nu}$ and dual Ricci scalar $\mathcal{R}$ by contracting indices of the dual Riemann tensor,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\alpha\nu\beta}\eta^{\alpha\beta} = \frac{1}{4}R^{\rho\sigma}{}_{\lambda\gamma}\epsilon_{\rho\sigma\mu\beta}\epsilon^{\lambda\gamma\alpha\beta}\eta_{\nu\alpha}, \tag{4.47}$$

$$\mathcal{R} = \mathcal{R}_{\mu\nu}\eta^{\mu\nu} = \frac{1}{4}R^{\rho\sigma}{}_{\lambda\gamma}\epsilon_{\rho\sigma\alpha\beta}\epsilon^{\lambda\gamma\alpha\beta}. \tag{4.48}$$

We note that there is also the 'simply' dual $\mathbf{R}_{\alpha\beta\mu\nu}$ in Equation (4.28) that dualizes only one of the antisymmetric pairs of the Riemann tensor. Scalars from this expression, such as the Lanczos $K_1 = \mathbf{R}^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, are redundant to what can be found with $\mathcal{R}_{\mu\nu\alpha\beta}$. Furthermore, the Ricci tensor and Ricci scalar duals from $\mathbf{R}_{\alpha\beta\mu\nu}$ are identically zero ($\mathbf{R}_{\mu\nu} = 0$, $\mathbf{R} = \mathbf{0}$), due to the first Bianchi identity.

Using the dual Riemann expressions in Equations (4.29), (4.47) and (4.48) the following identities can be derived by using the generalized Kronecker delta in Equations (4.32) and (4.33) on the combinations $\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and $\mathcal{R}^2$,

$$\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta} = \frac{1}{16}R_{\alpha\beta}{}^{\omega\tau}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\omega\tau\lambda\gamma}R^{\rho\sigma}{}_{\theta\phi}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\theta\phi\lambda\gamma} = \frac{1}{16}R_{\alpha\beta}{}^{\omega\tau}R^{\rho\sigma}{}_{\theta\phi}(-\delta^{\alpha\beta\mu\nu}_{\rho\sigma\mu\nu})(-\delta^{\theta\phi\lambda\gamma}_{\omega\tau\lambda\gamma}) = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \tag{4.49}$$

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} = \frac{1}{16}R^{\rho\sigma}{}_{\theta\gamma}\epsilon_{\rho\sigma\mu\beta}\epsilon^{\theta\gamma\alpha\beta}\eta_{\alpha\lambda}R_{\omega\tau}{}^{\xi\chi}\epsilon^{\omega\tau\mu\phi}\epsilon_{\xi\chi\delta\phi}\eta^{\delta\lambda} = \frac{1}{16}R^{\rho\sigma}{}_{\theta\gamma}R_{\omega\tau}{}^{\xi\chi}(-\delta^{\omega\tau\phi\mu}_{\rho\sigma\beta\mu})(-\delta^{\theta\gamma\beta\delta}_{\xi\chi\phi\delta}) = R_{\mu\nu}R^{\mu\nu}, \tag{4.50}$$

$$\mathcal{R}^2 = \frac{1}{16}R^{\rho\sigma}{}_{\lambda\gamma}\epsilon_{\rho\sigma\alpha\beta}\epsilon^{\lambda\gamma\alpha\beta}R^{\mu\nu}{}_{\omega\tau}\epsilon_{\mu\nu\theta\phi}\epsilon^{\omega\tau\theta\phi} = \frac{1}{16}R^{\rho\sigma}{}_{\lambda\gamma}R^{\mu\nu}{}_{\omega\tau}(-\delta^{\lambda\gamma\alpha\beta}_{\rho\sigma\alpha\beta})(-\delta^{\omega\tau\theta\phi}_{\mu\nu\theta\phi}). = R^2 \tag{4.51}$$

Similar to electrodynamics, each of the dual scalars can be expressed in terms of their corresponding original non-dual scalars. An equivalent Lagrangian for the linearized Gauss-Bonnet model $\mathcal{L} = \frac{1}{4}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$ can therefore be expressed as $\mathcal{L} = \frac{1}{4}(\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2)$. Writing this in dually symmetric form, similar to $\mathcal{L}_{MNH}$, the resulting Lagrangian density is,

$$\mathcal{L}_{LGB} = \frac{1}{8}(\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2 + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \tag{4.52}$$

This presentation would suggest analogy to the MNH equations. The above expression is invariant under interchange $R_{\mu\nu\alpha\beta} \leftrightarrow \mathcal{R}_{\mu\nu\alpha\beta}$, $R_{\mu\nu} \leftrightarrow \mathcal{R}_{\mu\nu}$ and $R \leftrightarrow \mathcal{R}$. However, recalling the invariant presented by Lanczos $K_2 = \mathcal{R}^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, who noticed the relationship $K_2 \propto I_3 - 4I_1 + I_2$, indeed deriving the identity for K_2 yields,

$$\mathcal{R}_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{1}{4}R^{\rho\sigma}{}_{\lambda\gamma}R_{\alpha\beta}{}^{\mu\nu}(-\delta^{\alpha\beta\lambda\gamma}_{\rho\sigma\mu\nu}) = -R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 4R_{\mu\nu}R^{\mu\nu} - R^2. \tag{4.53}$$

Using this identity the Lagrangian $\mathcal{L}_{LGB}$ can also be expressed as,

$$\mathcal{L}_{LGB} = -\frac{1}{4}\mathcal{R}_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{4.54}$$

This presentation seems to indicate an analogy with the MH equations. While the Lagrangian $\mathcal{L}_{LGB}$ can be expressed in dual form analogous to both halves of Maxwell's theory, this discrepancy will be clearly avoided for the equation of motion $E^{\rho\sigma}_{LGB}$ in the following sections, which corresponds to the second Bianchi identity as in the MH case. First the dualization of the energy-momentum tensor will be performed.

Dualizing the energy-momentum tensor

The energy-momentum tensor for the linearized Gauss-Bonnet gravity model, a well known expression given in Equation (4.27), was derived from Noether's theorem in [14]. To dualize this expression, a series of identities can be derived relating the terms in the energy-momentum tensor to the corresponding dual terms, as in the case of Equation (4.37). For the four terms in Equation (4.27) not proportional to Minkowski $(-R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu})$, the identities between dual and non-dual are, from Equations (4.29), (4.47) and (4.48) and using

Equations (4.32) and (4.33),

$$\mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma} = \frac{1}{16}\eta^{\nu\gamma}R_{\alpha\beta}{}^{\mu\xi}R_{\chi\delta}{}^{\theta\phi}(-\delta^{\chi\delta\lambda\sigma}_{\mu\xi\lambda\sigma})(-\delta^{\alpha\beta\omega\rho}_{\theta\phi\gamma\rho}) = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + \frac{1}{2}\eta^{\omega\nu}R_{\mu\gamma\alpha\beta}R^{\mu\gamma\alpha\beta}, \quad (4.55)$$

$$\mathcal{R}_{\rho\sigma}\mathcal{R}^{\omega\rho\nu\sigma} = \frac{1}{16}\eta^{\nu\gamma}R^{\mu\gamma}{}_{\beta\delta}R_{\theta\phi}{}^{\chi\xi}(-\delta^{\theta\phi\omega\rho}_{\mu\gamma\alpha\rho})(-\delta^{\beta\delta\alpha\sigma}_{\chi\xi\lambda\sigma}) = R_{\rho\sigma}R^{\omega\rho\nu\sigma} - R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + \frac{1}{4}\eta^{\omega\nu}R_{\mu\gamma\alpha\beta}R^{\mu\gamma\alpha\beta}, \quad (4.56)$$

$$\mathcal{R}^{\omega\lambda}\mathcal{R}^{\nu}{}_{\lambda} = \frac{1}{16}\eta^{\gamma\lambda}R_{\mu\tau}{}^{\rho\sigma}R_{\theta\phi}{}^{\chi\xi}(-\delta^{\mu\tau\omega\alpha}_{\rho\sigma\gamma\alpha})(-\delta^{\theta\phi\nu\beta}_{\chi\xi\lambda\beta}) = R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu} + \frac{1}{4}\eta^{\omega\nu}R^{2}, \quad (4.57)$$

$$\mathcal{R}\mathcal{R}^{\omega\nu} = \frac{1}{16}\eta^{\nu\gamma}R^{\mu\tau}{}_{\rho\sigma}R_{\theta\phi}{}^{\chi\xi}(-\delta^{\rho\sigma\alpha\beta}_{\mu\tau\alpha\beta})(-\delta^{\theta\phi\omega\lambda}_{\chi\xi\gamma\lambda}) = -RR^{\omega\nu} + \frac{1}{2}\eta^{\omega\nu}R^{2}. \quad (4.58)$$

Combining these 4 terms in the manner they appear in the energy-momentum tensor yields an interesting identity,

$$-\mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma} + 2\mathcal{R}_{\rho\sigma}\mathcal{R}^{\omega\rho\nu\sigma} + 2\mathcal{R}^{\omega\lambda}\mathcal{R}^{\nu}{}_{\lambda} - \mathcal{R}\mathcal{R}^{\omega\nu} = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu}, \quad (4.59)$$

as the particular coefficients of the energy-momentum tensor cancel all of the second and third terms in Equations (4.55) - (4.58). Since the term proportional to Minkowski can also be re-expressed $\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ as shown in Section 4.2.4, the energy-momentum tensor in Equation (4.27) can be expressed as $T^{\omega\nu}_{LGB} = -\mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma} + 2\mathcal{R}_{\rho\sigma}\mathcal{R}^{\omega\rho\nu\sigma} + 2\mathcal{R}^{\omega\lambda}\mathcal{R}^{\nu}{}_{\lambda} - \mathcal{R}\mathcal{R}^{\omega\nu} + \frac{1}{4}\eta^{\omega\nu}(\mathcal{R}_{\mu\gamma\alpha\beta}\mathcal{R}^{\mu\gamma\alpha\beta} - 4\mathcal{R}_{\mu\gamma}\mathcal{R}^{\mu\gamma} + \mathcal{R}^2)$. The goal of this section is to write the energy-momentum tensor in dually invariant form, thus a third equivalent representation based on the dual and non-dual is,

$$\begin{aligned} T^{\omega\nu}_{LGB} = &-\frac{1}{2}R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + R_{\rho\sigma}R^{\omega\rho\nu\sigma} + R^{\omega\lambda}R^{\nu}{}_{\lambda} - \frac{1}{2}RR^{\omega\nu} - \frac{1}{2}\mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma} + \mathcal{R}_{\rho\sigma}\mathcal{R}^{\omega\rho\nu\sigma} + \mathcal{R}^{\omega\lambda}\mathcal{R}^{\nu}{}_{\lambda} - \frac{1}{2}\mathcal{R}\mathcal{R}^{\omega\nu} \\ &+ \frac{1}{8}\eta^{\omega\nu}(R_{\mu\gamma\alpha\beta}R^{\mu\gamma\alpha\beta} - 4R_{\mu\gamma}R^{\mu\gamma} + R^2) + \frac{1}{8}\eta^{\omega\nu}(\mathcal{R}_{\mu\gamma\alpha\beta}\mathcal{R}^{\mu\gamma\alpha\beta} - 4\mathcal{R}_{\mu\gamma}\mathcal{R}^{\mu\gamma} + \mathcal{R}^2). \quad (4.60) \end{aligned}$$

This presentation of the energy-momentum tensor is explicitly invariant under interchange $R_{\mu\nu\alpha\beta} \leftrightarrow \mathcal{R}_{\mu\nu\alpha\beta}$, $R_{\mu\nu} \leftrightarrow \mathcal{R}_{\mu\nu}$ and $R \leftrightarrow \mathcal{R}$. This form, similar to the first $\mathcal{L}_{LGB}$ derived in Equation (4.52), is analogous to what is found for $T^{\omega\nu}_{MNH}$ in the MNH half of electrodynamics. Another dually invariant and equivalent expression can be considered by deriving an identity between the dual and non-dual tensors $\mathcal{R}^{\omega\alpha\beta\lambda}R^{\nu}{}_{\alpha\beta\lambda}$,

$$\mathcal{R}^{\omega\alpha\beta\lambda}R^{\nu}{}_{\alpha\beta\lambda} = \frac{1}{4}\eta^{\theta\nu}R_{\alpha\beta}{}^{\rho\sigma}R_{\theta\gamma}{}^{\mu\phi}(-\delta^{\alpha\beta\omega\gamma}_{\rho\sigma\mu\phi}) = -R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} + 2R_{\rho\sigma}R^{\omega\rho\nu\sigma} + 2R^{\omega\lambda}R^{\nu}{}_{\lambda} - RR^{\omega\nu}. \quad (4.61)$$

This is exactly the non-Minkowski part of Equation (4.27)! In addition, the Minkowski part in Equation (4.27) can be re-expressed using $\mathcal{R}^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = -R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 4R_{\mu\nu}R^{\mu\nu} - R^2$ from Equation (4.53), yielding a compact expression for the energy-momentum tensor of Gauss-Bonnet gravity in dual form,

$$T^{\omega\nu}_{LGB} = \mathcal{R}^{\omega\alpha\beta\lambda}R^{\nu}{}_{\alpha\beta\lambda} - \frac{1}{4}\eta^{\omega\nu}\mathcal{R}^{\mu\gamma\alpha\beta}R_{\mu\gamma\alpha\beta}. \quad (4.62)$$

In this presentation the energy-momentum tensor, similar to the second $\mathcal{L}_{LGB}$ derived in Equation (4.54), is analogous to what is found for $T^{\omega\nu}_{MH}$ in the MH half of electrodynamics. Both the Lagrangian $\mathcal{L}_{LGB}$ and energy-momentum tensor $T^{\omega\nu}_{LGB}$ can be expressed in dual form analogous to both halves on Maxwell's equations. It appears however that the homogenous half MH is truly analogous given compactness of these Equations (4.54) and (4.62), and the second Bianchi identity equation of motion. This will be evidenced by the equation of motion $E^{\rho\sigma}_{LGB}$ in dual form, which is the topic of the following section.

Dualizing the equation of motion

Conventional wisdom states that the *'In D = 4 the Gauss-Bonnet invariant is a total derivative, and hence does not contribute to the gravitational dynamics'* [91], and more specifically to the equation of motion, *'In the four-dimensional spacetime, the Gauss-Bonnet term does not contribute to the field equations since it becomes a total derivative'* [139]. This sentiment implies that there is simply nothing in the equation of motion following from the Gauss-Bonnet Lagrangian. A closer analysis shows that this is not the case. Differentiating the Gauss-Bonnet Lagrangian,

$$\frac{\partial\mathcal{L}_{LGB}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})} = \frac{1}{8}R_{\mu\nu\alpha\beta}[\epsilon^{\lambda\sigma\mu\nu}\epsilon^{\omega\rho\alpha\beta} + \epsilon^{\lambda\rho\mu\nu}\epsilon^{\omega\sigma\alpha\beta}]. \quad (4.63)$$

Substituting this into the Euler-Lagrange equation in Equation (4.20), using the $\partial_\omega\partial_\lambda$ and $R_{\mu\nu\alpha\beta}$ symmetries, and reintroducing the dual $\mathcal{R}^{\omega\rho\lambda\sigma}$ in Equation (4.29),

$$E^{\rho\sigma}_{LGB} = \partial_\omega\partial_\lambda\frac{\partial\mathcal{L}_{LGB}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})} = \frac{1}{4}\partial_\omega\partial_\lambda R_{\mu\nu\alpha\beta}\epsilon^{\lambda\mu\nu\sigma}\epsilon^{\omega\alpha\beta\rho} = \partial_\omega\partial_\lambda\mathcal{R}^{\omega\rho\lambda\sigma}. \quad (4.64)$$

The equation of motion for linearized Gauss-Bonnet gravity is based on the second order divergence of the dual Riemann tensor, analogous to how equation (4.40) is the divergence of the dual tensor for the homogenous half of electrodynamics. Similarly, using the Levi-Civita symbol, this can be re-expressed as the second Bianchi identity via $\partial_\lambda R_{\mu\nu\alpha\beta}\epsilon^{\lambda\mu\nu\sigma} = \frac{1}{3}\epsilon^{\lambda\mu\nu\sigma}(\partial_\lambda R_{\mu\nu\alpha\beta} + \partial_\mu R_{\nu\lambda\alpha\beta} + \partial_\nu R_{\lambda\mu\alpha\beta}) = 0$. Therefore the equation of motion for the linearized Gauss-Bonnet model can be expressed as the second Bianchi identity as,

$$E^{\rho\sigma}_{LGB} = \frac{1}{12}\epsilon^{\lambda\mu\nu\sigma}\epsilon^{\omega\alpha\beta\rho}\partial_\omega(\partial_\lambda R_{\mu\nu\alpha\beta} + \partial_\mu R_{\nu\lambda\alpha\beta} + \partial_\nu R_{\lambda\mu\alpha\beta}) = 0. \tag{4.65}$$

The linearized Gauss-Bonnet model can therefore be completely expressed in analogous dually invariant form to the homogenous half of electrodynamics (MH) in Section 4.2.3. The equation of motion for both of these models is the second Bianchi identity. This raises a point, perhaps of fundamental significance; if the Bianchi identity which represents half of Maxwell's equations is considered a fundamental equation of motion to electrodynamics, should the second Bianchi identity of the Riemann tensor be thought of as part of the fundamental set of equations for the Gauss-Bonnet theories of gravity, or more generally, metric theories of gravity? Such views have been considered in the literature in the past [122], but are not often included in the set of fundamental equations of motion as in the case of classical electrodynamics.

4.2.5 Completing the dual linearized gravity model

The linearized Gauss-Bonnet gravity model has been expressed in dual form, with $\mathcal{L}_{LGB}$ and $T^{\mu\nu}_{LGB}$ independently dual invariant analogous to the homogenous half of electrodynamics (MH) in Section 4.2.3. This analogy was further emphasized by the second Bianchi identity being the equation of motion for the model. One major issue arises here, however, in the fact that the equation of motion itself does not have a dual counterpart which can be found under interchange of the Riemann tensor $R_{\mu\nu\alpha\beta} \leftrightarrow \mathcal{R}_{\mu\nu\alpha\beta}$. This is a major issue for three reasons: (i) in order to introduce a dual equation of motion, another internally dual invariant must be introduced to the Lagrangian as in the MNH case, (ii) the possible Lagrangian densities ($\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$) are constrained by the procedure in [14], and (iii) the procedure in [14] showed that the Gauss-Bonnet energy-momentum tensor was the unique gauge invariant, symmetric and trace-free expression derived from Noether's theorem for these possible Lagrangian densities.

Problems (i) and (ii) can be easily remedied by noticing the dual equation of motion to $E^{\rho\sigma}_{LGB}$ is trivially of the form $\partial_\omega\partial_\lambda R^{\omega\rho\lambda\sigma}$, which follows from one of the constrained invariants $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, contraction of the linearized Riemann tensors. The model built from this scalar alone will now be explored. Reference to anything associated to the dual form of model derived from the linearized Riemann-Riemann scalar from here forward will have subscript LRR = linearized Riemann-Riemann for clarity. Therefore the current section explores the dual form of the Lagrangian density $\mathcal{L}_{LRR}$, equation of motion $E^{\rho\sigma}_{LRR}$, and energy-momentum tensor $T^{\mu\nu}_{LRR}$. Problem (iii) is significantly less trivial and will be discussed in detail.

Dualizing the Lagrangian and equation of motion

In order to have the equation of motion dual to $E^{\rho\sigma}_{LGB}$, a Lagrangian of the form $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ will be considered, namely $\mathcal{L}_{LRR} = -\frac{1}{4}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. From Equation (4.49), the dualization is trivial, since $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}$. Therefore it can equivalently be expressed as $\mathcal{L}_{LRR} = -\frac{1}{4}\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}$, and in dually symmetric form as,

$$\mathcal{L}_{LRR} = -\frac{1}{8}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}). \tag{4.66}$$

This dual Lagrangian is strikingly similar to that of the non-homogenous (MNH) half of electrodynamics. Differentiating this expression yields $\frac{\partial\mathcal{L}_{LRR}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})} = \frac{1}{2}[R^{\omega\rho\lambda\sigma} + R^{\lambda\rho\omega\sigma}]$. The Euler-Lagrange equation of motion from Equation (4.20) is therefore,

$$E^{\rho\sigma}_{LRR} = \partial_\omega\partial_\lambda\frac{\partial\mathcal{L}_{LRR}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})} = \partial_\omega\partial_\lambda R^{\omega\rho\lambda\sigma}, \tag{4.67}$$

which is indeed the expression dual to $E^{\rho\sigma}_{LGB}$. The LRR model therefore has an internally dual symmetric Lagrangian, and an equation of motion dual to that of linearized Gauss-Bonnet gravity; both which are analogous to the non-homogenous half of electrodynamics. Problem (iii) is now to derive the energy-momentum tensor, which is not trivially gauge invariant as $T^{\mu\nu}_{LGB}$ is in [14].

Dualizing the energy-momentum tensor

In [1] the linearized Gauss-Bonnet gravity model was the unique model derived from the procedure for $N = M = 2$ and had a gauge invariant energy-momentum tensor. To understand why this is, we must consider the conserved current from Noether's first theorem in Equation (4.20) for a Lagrangian density of the form $\partial\partial h\partial\partial h$,

$$\partial_\omega\left[\frac{\partial\mathcal{L}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})}\partial_\lambda\delta h_{\rho\sigma} + \eta^{\omega\nu}\mathcal{L}\delta x_\nu - \left(\partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\omega\partial_\lambda h_{\rho\sigma})}\right)\delta h_{\rho\sigma}\right] = 0. \tag{4.68}$$

The third term above is responsible for the lack of gauge invariance in the model, since the transformation $\delta h_{\rho\sigma} = -2\Gamma^\nu{}_{\rho\sigma}\delta x_\nu$ [14, 26, 110], where $\Gamma^\nu{}_{\rho\sigma} = \frac{1}{2}(\partial^\nu h_{\rho\sigma} - \partial_\rho h^\nu_\sigma - \partial_\sigma h^\nu_\rho)$, is not gauge invariant. This is in essence the same reason for the no-go result that spin-2 linearized gravity cannot have a gauge invariant energy-momentum tensor [14, 163, 140, 61, 63], at least second order derivatives are needed. Only for the linearized Gauss-Bonnet Lagrangian density does the particular combination of invariants kill this term, resulting in a gauge invariant expression. The first term is gauge invariant because of $\partial_\lambda \delta h_{\rho\sigma}$ yielding the linearized Riemann tensor via $R^\nu{}_{\rho\sigma\lambda} = \partial_\lambda \Gamma^\nu_{\rho\sigma} - \partial_\sigma \Gamma^\nu_{\rho\lambda}$ which is independently gauge invariant. To show this explicitly, deriving the energy-momentum tensor from $\mathcal{L}_{LRR}$ in Equation (4.68),

$$T^{\omega\nu}_{LRR} = R^{\omega\rho\lambda\sigma} R^\nu{}_{\rho\lambda\sigma} - \frac{1}{4}\eta^{\omega\nu} R_{\mu\gamma\alpha\beta} R^{\mu\gamma\alpha\beta} - 2\partial_\lambda R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma}. \tag{4.69}$$

This expression is not dually symmetric, not gauge invariant and not trace-free. It is merely conserved on-shell via $E^{\rho\sigma}_{LRR}$. This feature is expected since it is related to the main result of [14], however it greatly hampers the development of a complete dual formulation. There are only two possible remedies to this problem. One is to integrate by parts the third term in the energy-momentum tensor $T^{\omega\nu}_{LRR}$, since one term will combine with the first time in $T^{\omega\nu}_{LRR}$ and the other term will be a second order term of the form $\partial_\omega \partial_\lambda (R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma})$. What remains under ∂_ω is $2R^{\omega\rho\lambda\sigma} R^\nu{}_{\rho\lambda\sigma} - \frac{1}{4}\eta^{\omega\nu} R_{\mu\gamma\alpha\beta} R^{\mu\gamma\alpha\beta}$ which is indeed gauge invariant and symmetric, but is neither trace-free nor conserved. Additionally the second order term does not trivially vanish because the symmetries $\omega\lambda$ and $\rho\sigma$ are unable to yield the first Bianchi identity.

The second, and more reasonable solution, is to integrate by parts the equation of motion. This is possible because Noether's first theorem is used to derive a complete identity given in Equation (4.20), it is not simply a method of deriving equations of motion and conservation laws separately. For LRR this yields $-2E^{\rho\sigma}_{LRR} \Gamma^\nu{}_{\rho\sigma} + \partial_\omega T^{\omega\nu}_{LRR} = 0$, which expands to,

$$2\partial_\omega \partial_\lambda R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma} + \partial_\omega (R^{\omega\rho\lambda\sigma} R^\nu{}_{\rho\lambda\sigma} - \frac{1}{4}\eta^{\omega\nu} R_{\mu\gamma\alpha\beta} R^{\mu\gamma\alpha\beta} - 2\partial_\lambda R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma}) = 0. \tag{4.70}$$

Integration by parts of the first term (equation of motion) via $2\partial_\omega \partial_\lambda R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma} = 2\partial_\omega [\partial_\lambda R^{\omega\rho\lambda\sigma} \Gamma^\nu{}_{\rho\sigma}] - 2\partial_\lambda R^{\omega\rho\lambda\sigma} \partial_\omega \Gamma^\nu{}_{\rho\sigma}$ exactly kills the third term in the energy-momentum expression. What is left in place of the equation of motion is $-2\partial_\lambda R^{\omega\rho\lambda\sigma} \partial_\omega \Gamma^\nu{}_{\rho\sigma} = -(\partial_\lambda R^{\omega\rho\lambda\sigma}) R^\nu{}_{\rho\omega\sigma}$, therefore we have the relationship between this expression and the total divergence from Noether's first theorem as,

$$(\partial_\lambda R^{\omega\rho\lambda\sigma}) R^\nu{}_{\rho\omega\sigma} = \partial_\omega (R^{\omega\rho\lambda\sigma} R^\nu{}_{\rho\lambda\sigma} - \frac{1}{4}\eta^{\omega\nu} R_{\mu\gamma\alpha\beta} R^{\mu\gamma\alpha\beta}). \tag{4.71}$$

What is left in the divergence, is a precisely gauge invariant, symmetric and trace-free energy-momentum tensor! This expression is conserved on-shell via the equation $\bar{E}^{\rho\lambda\sigma}_{DRR} = \partial_\lambda R^{\omega\rho\lambda\sigma}$. This equation of motion can be expressed in dual form with the Gauss-Bonnet equation of motion, since the Gauss-Bonnet equation of motion is the second Bianchi identity which requires only one derivative of the dual Riemann tensor $\partial_\lambda \mathcal{R}^{\omega\rho\lambda\sigma}$, therefore not impacting the Lagrangian $\mathcal{L}_{LGB}$ or energy-momentum tensor $T^{\omega\nu}_{LGB}$ derived in [14]. Furthermore, this energy-momentum tensor can be expressed in dually symmetric form analogous to the non-homogenous half of electrodynamics.

Since this presentation differs from that to this point (limiting the equation of motion to a single divergence of the field strength tensors), the dual form of the Riemann-Riemann model will now be referred to with subscript DRR for clarity. The associated Gauss-Bonnet model will be referred to with subscript DGB for clarity. Their equations of motion are a single divergence of the dual and non-dual Riemann tensor, forming a dually invariant pair of equations of motion $\bar{E}^{\rho\lambda\sigma}_{DRR} = \partial_\lambda R^{\omega\rho\lambda\sigma}$ and $\bar{E}^{\rho\lambda\sigma}_{DGB} = \partial_\lambda \mathcal{R}^{\omega\rho\lambda\sigma}$, where the bar represents that the single divergence equation of motion follows from integration by parts of the Euler-Lagrange equation necessary for a gauge invariant, conserved, symmetric and trace-free energy-momentum tensor. These equations of motion correspond to the Maxwell-like higher spin gauge theories for $N = M = 2$, models that have been well explored in the literature [82, 79, 21]. The dually invariant Lagrangian densities remain unchanged $\mathcal{L}_{DRR} = -\frac{1}{8}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta})$ and $\mathcal{L}_{DGB} = -\frac{1}{4}\mathcal{R}_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. The energy-momentum tensor $T^{\omega\nu}_{DRR} = R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} - \frac{1}{4}\eta^{\omega\nu}R_{\mu\gamma\alpha\beta}R^{\mu\gamma\alpha\beta}$ can also be expressed in dually symmetric form by re-writing the term proportional to Minkowski via the identity in Equation (4.55) ($-\frac{1}{4}\eta^{\omega\nu}R_{\mu\gamma\alpha\beta}R^{\mu\gamma\alpha\beta} = -\frac{1}{2}R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} - \frac{1}{2}\mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma}$), which yields,

$$T^{\omega\nu}_{DRR} = \frac{1}{2}(R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} - \mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma}), \tag{4.72}$$

which is analogous to the MNH energy-momentum tensor $T^{\mu\nu}_{MNH}$ in Equation (4.38). Therefore the linearized gravity model can be expressed completely in dual invariant form by considering the equation of motion dual to the linearized Gauss-Bonnet equation of motion, namely the second Bianchi identity; thus the linearized gravity model from $\mathcal{L}_{DGB}$ and $\mathcal{L}_{DRR}$ is analogous to the complete theory of electrodynamics. The equations of motion consisting of a single derivative of the linearized Riemann and dual Riemann tensors is a consequence of the requirement that the models be completely gauge invariant under the spin-2 gauge transformation (linearized diffeomorphisms), as well as have energy-momentum tensors that are conserved, symmetric and trace-free. The $N = M = 1$ and $N = M = 2$ dual formulations will be summarized in the following section.

4.2.6 Complete dual models for $N = M = 1$ and $N = M = 2$

The two models derived in [14], namely electrodynamics and the linearized Gauss-Bonnet model have been expressed in dual form. This involved considering the general Lagrangian density $\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$ derived in [14] for $N = M = 2$, of which $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ (DRR) is required to have an equation of motion dual to the second Bianchi identity, which was derived from the Euler-Lagrange equation of the linearized Gauss-Bonnet model (DGB). The electrodynamics model derived from $N = M = 1$, consisting of the complete set of Maxwell's equations in the Euler-Lagrange equation, was expressed in terms of the conventional Lagrangian which corresponds to the non-homogenous set of Maxwell's equations (MNH), and the Lagrangian which corresponds to the homogenous set of Maxwell's equation (MH). A summary of the models for (MH), (MNH), (DGB) and (DRR) is presented below,

Equation	**Model** $N = M = 1$	**Model** $N = M = 2$
Non-Homogenous E^A	$E^\rho_{MNH} = \partial_\sigma F^{\sigma\rho}$	$\bar{E}^{\rho\lambda\sigma}_{DRR} = \partial_\lambda R^{\omega\rho\lambda\sigma}$
Homogenous E^A (I)	$E^\rho_{MH} = \partial_\sigma \mathcal{F}^{\sigma\rho}$	$\bar{E}^{\rho\lambda\sigma}_{DGB} = \partial_\lambda \mathcal{R}^{\omega\rho\lambda\sigma}$
Homogenous E^A (II)	$\partial_\sigma F_{\alpha\beta} + \partial_\beta F_{\sigma\alpha} + \partial_\alpha F_{\beta\sigma} = 0$	$\partial_\omega R_{\mu\nu\alpha\beta} + \partial_\mu R_{\nu\omega\alpha\beta} + \partial_\nu R_{\omega\mu\alpha\beta} = 0$
Non-Homogenous $\mathcal{L}$	$\mathcal{L}_{MNH} = -\frac{1}{8}(F_{\mu\nu}F^{\mu\nu} - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})$	$\mathcal{L}_{DRR} = -\frac{1}{8}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta})$
Homogenous $\mathcal{L}$	$\mathcal{L}_{MH} = -\frac{1}{4}\mathcal{F}_{\mu\nu}F^{\mu\nu}$	$\mathcal{L}_{DGB} = -\frac{1}{4}\mathcal{R}_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$
Non-Homogenous $T^{\omega\nu}$	$T^{\omega\nu}_{MNH} = \frac{1}{2}[F^{\omega\alpha}F^{\nu}{}_{\alpha} + \mathcal{F}^{\omega\alpha}\mathcal{F}^{\nu}{}_{\alpha}]$	$T^{\omega\nu}_{DRR} = \frac{1}{2}[R^{\omega\rho\lambda\sigma}R^{\nu}{}_{\rho\lambda\sigma} - \mathcal{R}^{\omega\rho\lambda\sigma}\mathcal{R}^{\nu}{}_{\rho\lambda\sigma}]$
Homogenous $T^{\omega\nu}$	$T^{\omega\nu}_{MH} = \mathcal{F}^{\omega\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\omega\nu}\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$	$T^{\omega\nu}_{DGB} = \mathcal{R}^{\omega\alpha\beta\lambda}R^{\nu}{}_{\alpha\beta\lambda} - \frac{1}{4}\eta^{\omega\nu}\mathcal{R}^{\mu\gamma\alpha\beta}R_{\mu\gamma\alpha\beta}$

where E^A represents a general Euler-Lagrange equation of motion. Note that the sign change in the Lagrangian and energy-momentum tensor from $N = M = 1$ to $N = M = 2$ is a result of the Minkowski metric causing negative generalized Kronecker deltas, of which $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ terms have one and $\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\beta}$ terms have two such contributions. The homogenous equations of motion have been expressed both (I) in terms of the dual field strength tensor and (II) in terms of the expanded form (second Bianchi identity). From the summary above, every Lagrangian, equation of motion and energy-momentum tensor is completely and independently invariant under the gauge transformations $A'_\mu = A_\mu + \partial_\mu\phi$ and $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, thus complete gauge invariance derived in [14] is maintained. Each Lagrangian density and energy-momentum tensor can be expressed in independently dual invariant form. Each model has a pair of dually invariant equations of motion, namely Maxwell's equations $E^\rho_{MH} \leftrightarrow E^\rho_{MNH}$ for $N = M = 1$ and the linearized gravity model $\bar{E}^{\rho\lambda\sigma}_{DRR} \leftrightarrow \bar{E}^{\rho\lambda\sigma}_{DRR}$ for $N = M = 2$. The two homogenous halves of the models (MH and DBG) have common form between Lagranians $\mathcal{L}_{MH}$, $\mathcal{L}_{DGB}$, equations of motion E^ρ_{MH} , $\bar{E}^{\rho\lambda\sigma}_{DGB}$, and energy-momentum tensors $T^{\omega\nu}_{MH}$, $T^{\omega\nu}_{DGB}$. Similarly, the two non-homogenous halves of the models (MNH and DRR) have common form between Lagrangians $\mathcal{L}_{MNH}$, $\mathcal{L}_{DRR}$, equations of motion E^ρ_{MNH} , $\bar{E}^{\rho\lambda\sigma}_{DRR}$, and energy-momentum tensors $T^{\omega\nu}_{MNH}$, $T^{\omega\nu}_{DRR}$. Due to the common form of the two models we can generalize the complete gauge invariant dual formulations to an arbitrary field strength tensor S for $N = M = n$, with equations of motion corresponding to the Maxwell-like higher spin gauge theories [82, 79, 21], which will be presented in the following section.

4.2.7 Generalization to Maxwell-like higher spin gauge theories

Gauge invariant curvature (field strength) tensors have long been generalized to all spin-n models, representing the case $N = M = n$. The models for $N = M = 1$ and $N = M = 2$ are built using the spin-1 (Maxwell field strength $F^{\omega\alpha}$) and spin-2 (linearized Riemann $R^{\mu\nu\alpha\beta}$) curvature tensors. For example, in the spin-3 and spin-4 cases, respectively, we have the field strength (curvature) tensors [56, 181],

$$\begin{aligned} S^{[\tau\nu][\kappa m][\chi\gamma]} = {} & \partial^\chi\partial^\tau\partial^\kappa\phi^{\gamma m\nu} + \partial^\tau\partial^m\partial^\gamma\phi^{\kappa\chi\nu} + \partial^\kappa\partial^\nu\partial^\gamma\phi^{\chi\tau m} + \partial^\chi\partial^m\partial^\nu\phi^{\kappa\tau\gamma} \\ & - \partial^\gamma\partial^m\partial^\nu\phi^{\chi\tau\kappa} - \partial^\chi\partial^\tau\partial^m\phi^{\kappa\nu\gamma} - \partial^\kappa\partial^\chi\partial^\nu\phi^{\tau m\gamma} - \partial^\tau\partial^\kappa\partial^\gamma\phi^{\chi m\nu}, \end{aligned} \tag{4.73}$$

$$\begin{aligned} S^{[ab][\gamma\chi][m\kappa][\nu\tau]} = {} & \partial^a\partial^\gamma\partial^m\partial^\nu\phi^{b\chi\kappa\tau} + \partial^a\partial^\gamma\partial^\kappa\partial^\tau\phi^{b\chi m\nu} + \partial^a\partial^m\partial^\chi\partial^\tau\phi^{b\kappa\gamma\nu} + \partial^a\partial^\nu\partial^\chi\partial^\kappa\phi^{b\tau\gamma m} \\ & + \partial^\gamma\partial^m\partial^b\partial^\tau\phi^{\chi\kappa a\nu} + \partial^\gamma\partial^\nu\partial^b\partial^\kappa\phi^{\chi\tau a m} + \partial^m\partial^\nu\partial^b\partial^\chi\phi^{\kappa\tau a\gamma} + \partial^b\partial^\chi\partial^\kappa\partial^\tau\phi^{a\gamma m\nu} \\ & - \partial^a\partial^\gamma\partial^m\partial^\tau\phi^{b\chi\kappa\nu} - \partial^a\partial^\gamma\partial^\nu\partial^\kappa\phi^{b\chi\tau m} - \partial^a\partial^m\partial^\nu\partial^\chi\phi^{b\kappa\tau\gamma} - \partial^\gamma\partial^m\partial^\nu\partial^b\phi^{\chi\kappa\tau a} \\ & - \partial^a\partial^\chi\partial^\kappa\partial^\tau\phi^{b\gamma m\nu} - \partial^\gamma\partial^b\partial^\kappa\partial^\tau\phi^{\chi a m\nu} - \partial^m\partial^b\partial^\chi\partial^\tau\phi^{\kappa a\gamma\nu} - \partial^\nu\partial^b\partial^\chi\partial^\kappa\phi^{\tau a\gamma m}. \end{aligned} \tag{4.74}$$

These tensors are generalizations of the linearized Riemann tensors with n pairs of antisymmetric indices that are symmetric under interchange, with totally symmetric ϕ for all n. They are exactly invariant under the spin-n gauge transformations [181]. For spin-3 and spin-4, respectively, these gauge transformations are,

$$\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a\lambda_{b\rho} + \partial_b\lambda_{a\rho} + \partial_\rho\lambda_{ab}, \tag{4.75}$$

$$\phi'_{b\chi\kappa\tau} = \phi_{b\chi\kappa\tau} + \partial_b\lambda_{\chi\kappa\tau} + \partial_\chi\lambda_{b\kappa\tau} + \partial_\kappa\lambda_{b\chi\tau} + \partial_\tau\lambda_{b\chi\kappa}, \tag{4.76}$$

where the gauge parameters λ are totally symmetric for all n. The general form of the equations of motion in the previous section are a single divergence of a curvature tensor; models that have been well worked out in the literature for the spin-n curvature tensors, known as the Maxwell-like higher spin gauge theories [82, 79, 21]. The analogy comes from the single divergence of the spin-1 curvature tensor in the case of electromagnetic theory. The dual formulation of these higher spin models has already been explored to some degree [19, 104]. In addition, scalars built from the contraction of these higher spin curvature tensors have been considered as exactly gauge invariant Lagrangian densities [78]. Using the curvature tensors of higher spin gauge theories we therefore can build completely dual and gauge invariant models in the analogous form of the $N = M = 1$ and $N = M = 2$ cases summarized in Section 4.2.6. The general form for the homogenous H and non-homogenous NH equations can be expressed using a general spin-n field strength tensor S and its dual $\mathcal{S}$, suppressing contracted indices, as,

Equation	**Model** $N = M = n$
Non-Homogenous E^A	$E^A_{NH} = \partial S^A$
Homogenous E^A	$E^A_H = \partial \mathcal{S}^A$
Non-Homogenous $\mathcal{L}$	$\mathcal{L}_{NH} = -\frac{1}{8}(SS \pm \mathcal{S}\mathcal{S})$
Homogenous $\mathcal{L}$	$\mathcal{L}_H = -\frac{1}{4}S\mathcal{S}$
Non-Homogenous $T^{\omega\nu}$	$T^{ab}_{NH} = \frac{1}{2}[S^{\omega}S^{\nu} \mp \mathcal{S}^{\omega}\mathcal{S}^{\nu}]$
Homogenous $T^{\omega\nu}$	$T^{ab}_H = \mathcal{S}^{\omega}S^{\nu} - \frac{1}{4}\eta^{\omega\nu}\mathcal{S}S$

where the $\pm$ and $\mp$ refer to odd n models (top sign) and even n models (bottom sign) due to the generalized Kronecker delta in Minkowski spacetime. It is worth noting that the four invariants for $N = M = 1$ and $N = M = 2$ are associated to (omitting indices) the wedge product between the differential forms representing the field strength tensor of electrodynamics F and its dual $\star F$, and the differential forms representing the Riemann tensor R and its dual $\star R$. Roughly speaking these correspond to the 4 invariants as $\mathcal{L}_{MH} \propto F \wedge F$, $\mathcal{L}_{MNH} \propto F \wedge \star F$, $\mathcal{L}_{DGB} \propto R \wedge R$ and $\mathcal{L}_{DRR} \propto R \wedge \star R$. In general the conjecture can be made that the higher spin models for $N = M = n$ will consider of invariants of field strength S, its dual $\mathcal{S}$, and differential form S, the presumed general form of the homogenous H and non-homogenous NH invariants will be $\mathcal{L}_H \propto S \wedge S$ and $\mathcal{L}_{NH} \propto S \wedge \star S$.

If we are to consider the combined action $\mathcal{L}_H + \mathcal{L}_{NH} = -\frac{1}{4}S\mathcal{S} - \frac{1}{8}(SS \pm \mathcal{S}\mathcal{S})$ for a general model, there are two possible generalizations that can be noted. First is the possibility to have a linear combination of the field strength and dual factored $\mathcal{L}_H + \mathcal{L}_{NH} \approx \pm\frac{1}{8}(S + \mathcal{S})(S + \mathcal{S})$, with signs depending on the particular S. The second is that, in the case of electrodynamics where we know the specific fields in each component, this generalization produces $\mathcal{L}_{MH} + \mathcal{L}_{MNH} \approx \frac{1}{2}B^2 - \frac{1}{2}E^2 + \vec{B} \cdot \vec{E}$. These two definitions are strikingly similar to the law of cosines where $c^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = a^2 + b^2 - 2\vec{a} \cdot \vec{b}$. This similarly, if any meaningful relationship exists, has not been determined.

4.2.8 Conclusions

In [14], a procedure was developed for building completely gauge invariant models by imposing gauge invariance and using Noether's first theorem for general Lagrangian densities of N order of derivatives and M rank of tensor potential. For $N = M = 1$ electrodynamics was uniquely derived, and for $N = M = 2$ linearized Gauss-Bonnet gravity was uniquely derived. Both of these models have the property of complete gauge invariance of the Lagrangian, equation of motion and energy-momentum tensor. The energy momentum tensors are gauge invariant, symmetric, trace-free and conserved.

In order to further investigate this relationship, electrodynamics and linearized Gauss-Bonnet gravity were expressed in their respective dual formulations. The Gauss-Bonnet model, conventionally claimed to have simply no equation of motion, in fact has the second Bianchi identity as its equation of motion, analogous to the homogenous half of Maxwell's equations in electrodynamics. In order to introduce the equation of motion dual to this expression, the linearized Riemann-Riemann Lagrangian was introduced, whose dual formulation was analogous to the non-homogenous half of electrodynamics. In this presentation, the electrodynamics and linearized gravity models have internal dual symmetries in their Lagrangians and energy-momentum tensors, and have equations of motion that are dual between the homogenous and non-homogenous halves of the models. The energy-momentum tensors are all gauge invariant, symmetric, trace-free and conserved, with the additional property of dual invariance being explicit in this formulation.

The dual formulation shared by these two models allows for their homogenous and non-homogenous halves to be expressed in a more general framework. The equations of motion of this general framework correspond to the Maxwell-like higher spin gauge theories built from the spin-n curvature tensors. These models are completely gauge invariant in the same manner as the electrodynamics and linearized Gauss-Bonnet gravity cases. In addition they are dually invariant analogous to the results in this article. Obtaining physical models which have some uniqueness criteria that separate themselves from other possible equations has been one of the focuses of theoretical physics in recent decades. Electrodynamics, perhaps the most successful model in physics, has a plethora of such properties: complete gauge invariance, conformal invariance, dual invariance, a trace-free and symmetric energy-momentum tensor, just to name a few. What we have shown is that these uniqueness properties can be generalized to the higher spin (Maxwell-like) gauge theories, where the linearized Gauss-Bonnet gravity model is the $N = M = 2$ analogue to the homogenous half of Maxwell's equations. Recent research has brought great renewed interest in the Gauss-Bonnet gravity model [91], as it has been claimed to provide 'new' dynamical predictions that explain astronomical observations to a higher degree of accuracy. Due to this, the $N = M = 2$ model can be applied to some of these observations to see if it too can better explain some observed phenomena; this application is the subject of future work.

4.3 Curvature tensors of higher-spin gauge theories derived from general Lagrangian densities

Abstract Curvature tensors of higher-spin gauge theories have been known for some time. In the past, they were postulated using a generalization of the symmetry properties of the Riemann tensor (curl on each index of a totally symmetric rank-n field for each spin-n). For this reason they are sometimes referred to as the generalized 'Riemann' tensors. In this article, a method for deriving these curvature tensors from first principles is presented; the derivation is completed without any a priori knowledge of the existence of the Riemann tensors or the curvature tensors of higher-spin gauge theories. To perform this derivation, a recently developed procedure for deriving exactly gauge invariant Lagrangian densities from quadratic combinations of N order of derivatives and M rank of tensor potential is applied to the $N = M = n$ case under the spin-n gauge transformations. This procedure uniquely yields the Lagrangian for classical electrodynamics in the $N = M = 1$ case and the Lagrangian for higher derivative gravity ('Riemann' and 'Ricci' squared terms) in the $N = M = 2$ case. It is proven here by direct calculation for the $N = M = 3$ case that the unique solution to this procedure is the spin-3 curvature tensor and its contractions. The spin-4 curvature tensor is also uniquely derived for the $N = M = 4$ case. In other words, it is proven here that, for the most general linear combination of scalars built from N derivatives and M rank of tensor potential, up to $N = M = 4$, there exists a unique solution to the resulting system of linear equations as the contracted spin-n curvature tensors. Conjectures regarding the solutions to the higher spin-n $N = M = n$ are discussed.

4.3.1 Motivation

Higher-spin gauge theories describing free massless fields are well established in the literature. These theories have gauge transformations and curvature tensors that have been generalized for any spin-n model considered [83, 72, 61, 56, 181]. In the past the curvature tensors were postulated based on symmetry properties of the Riemann tensors (by taking the curl on each index of a totally symmetric rank-n field for each spin-n [56]). Here we present a method to derive these curvature tensors from first principles; they are derived by direct calculation without any knowledge of the existence of the Riemann tensors or curvature tensors of higher-spin gauge theories.

The higher-spin curvature tensors, sometimes referred to as the generalized 'Riemann' curvature tensors for their generalization as n pairs of antisymmetric indices for each spin-n model analogous to the Riemann tensor in the $n = 2$ case, are independently gauge invariant under the spin-n gauge transformations. They are of particular interest in the Maxwell-like higher spin models that consider equations built from the divergence of these curvature tensors, which are analogous to Maxwell's equation in the spin-1 case. The Maxwell-like higher spin models have been primarily worked out in a series of papers by Francia et al. [79, 80, 21, 81]. The curvature tensors also allow for the generalization of the dual formulation of higher spin models used commonly for models built with the $n = 1$ field strength tensor $F^{\mu\nu}$ and $n = 2$ Riemann tensor $R^{\mu\nu\alpha\beta}$ [103, 57]. Generalization of the curvature tensors can be found in [181, 20]. In the past these generalizations have been developed by extrapolating from the symmetries of lower spin-n models, rather than by derivation from some general principles. The latter approach is what we will develop in this article: for each spin-n model, we will independently and uniquely derive the curvature tensors and their contractions (the 'Ricci' forms of the curvature tensors) from a general linear system of scalars, without any a priori knowledge of their existence. No knowledge of the curvature tensors or required symmetries is necessary for this procedure; only the form of the spin-n gauge transformations given in equations (4.77) to (4.80) is needed to perform this derivation.

Recent research developed a procedure for deriving completely gauge invariant Lagrangians by considering general linear systems of scalars under a particular gauge transformation [14]. The general Lagrangian density is expressed in terms of free coefficients which are solved for such that the resulting Lagrangian density is exactly gauge invariant (not only invariant up to a surface term). The scalars are built from quadratic combinations of N order of derivatives of M rank of tensor potentials. When this procedure is applied to the $N = M = 1$ case and the spin-1 gauge transformation $A'_\mu = A_\mu + \partial_\mu \xi$ is used, the Lagrangian $\mathcal{L} = CF_{\mu\nu}F^{\mu\nu}$ is uniquely derived. When it is applied to the $N = M = 2$ case and a spin-2 gauge transformation (linearized diffeomorphism) $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ is used, the Lagrangian $\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$ is uniquely derived.

The natural question that arose was what would occur if this procedure were applied to the $N = M = n$ case. Since the $N = M = 1$ case yields the scalar $F_{\mu\nu}F^{\mu\nu}$, built from the field strength tensor of electrodynamics, and the $N = M = 2$ yields the scalars $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$ and R^2, built from the linearized Riemann tensor and its contractions, it was conjectured that to derive an exactly gauge invariant Lagrangian from this procedure, it would be necessary to have contraction of independently gauge invariant 'field strength' (curvature) tensors. The validity of this conjecture is further investigated in this article.

To explore the extension to $N = M = n$, we started with $N = M = 3$ under the spin-3 gauge transformation $\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a \lambda_{b\rho} + \partial_b \lambda_{a\rho} + \partial_\rho \lambda_{ab}$, where $\lambda_{b\rho}$ is a symmetric gauge parameter. As in [14], we will only consider totally symmetric fields ϕ in this article, however in principle this procedure should work for any field symmetries, such as antisymmetric field models [148]. In this article we will show that, as it did for spin-1 and spin-2 in [14], the procedure yields a solution of several scalar invariants, which turn out to be the contracted spin-3 curvature tensor, $K^{\tau\nu\kappa m\chi\gamma}$ [20] (sometimes referred to as the 'Riemann' tensor generalization), and its 'Ricci' tensors, $K^{\tau\nu\kappa m}$ and $K^{\tau\nu}$. In other words the higher spin curvature tensors can be derived from this procedure without a priori knowledge of their existence. Extending this to $N = M = 4$ again yields the contraction of the independently gauge invariant curvature tensor $K^{ab\gamma\chi m\kappa\nu\tau}$, namely the spin-4 generalization of the 'Riemann' curvature tensor. For $N = M = 5$ and greater, the calculations became too difficult for us to do by hand (since each had thousands of scalar terms in the general expression), so, instead, conjectures about the nature of the spin-n Lagrangians, based on the cases $n = 1, 2, 3, 4$, are made at the end of the article. We note that higher spin models [149, 141] and the Lagrangian formulation for higher spin models is well researched from various points of view [59, 77, 38], but primarily these consider conventional (second order) spin-n equations of motion. Here, when we consider $N = M = n$ models, we have terms in the Lagrangian that are quadratic combinations of n order of derivatives and n rank of tensor potential. These types of higher spin Lagrangians are less developed in the literature [78]. For our purposes, we use these Lagrangians to derive the curvature tensors of higher-spin gauge theories without any a priori knowledge of their existence.

We acknowledge that there are several issues related to the unitary and renormalizability of higher derivative theories, that continue to be worked out in the literature [2, 35]. At no point do we consider the higher derivative Lagrangian densities we write down for higher $N = M = n$ cases to avoid or solve these problems: our motivation is purely to give a derivation of the well known curvature tensors of higher spin gauge theories without a priori knowledge of their existence, or of existence of the Riemannian tensors. We do this because previously they have been merely postulated using a generalization of the symmetry properties of the Riemann tensor (curl on each index of a totally symmetric rank-n field for each spin-n). Our method more naturally and independently obtains them alongside the Riemannian tensors and electrodynamics field strength tensor, since these tensors are the natural outcomes for the $N = M = 2$ and $N = M = 1$ derivations, respectively. The models associated to the Lagrangian densities we use to derive the higher spin curvature tensors have no, to our knowledge, new predictive insight of physical phenomena.

The article will be structured as follows. First, we will detail the procedure for deriving completely gauge invariant models by considering the spin-1 case and discuss generalizations of the gauge transformations to spin-n. Next, we will start from the general Lagrangian for the $N = M = 3$ case and, under the spin-3 gauge transformation, show how this yields precisely the contractions of the 'Riemann' and 'Ricci' curvature tensors of spin-3. This process will then be repeated for spin-4. Finally, we will provide conjectures about the behaviour of Lagrangians derived from the procedure for $N = M = n$, giving some indication of how these Lagrangians will be built for the spin-n case.

4.3.2 Derivation of the curvature tensors of higher-spin gauge theories

In [14], a procedure for deriving exactly gauge invariant Lagrangians is outlined in detail for the case $N = M = 1$ and $N = M = 2$. This procedure involves writing down the most general scalars for each case and solving for free coefficients such that the resulting Lagrangian density is exactly invariant under the gauge transformation being considered. For the $N = M = 1$ case, for which the most general scalar is the sum of all possible scalars of the form $\partial A \partial A$, when the spin-1 gauge transformation $A'_\mu = A_\mu + \partial_\mu \xi$ is applied, the resulting Lagrangian is $\mathcal{L} = CF_{\mu\nu}F^{\mu\nu}$. For the $N = M = 2$ case, for which the most general scalar is the sum of all possible scalars of the form $\partial\partial h \partial\partial h$, when the spin-2 gauge transformation $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ is applied, the resulting Lagrangian decouples into three independently gauge invariant scalars that turn out to be the linearized 'Riemann' and 'Ricci' tensors; $\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$. The obvious next step is to generalize the procedure for the $N = M = n$ case. This generalization, as we will show, can be used to derive the curvature tensors of higher-spin gauge thoeries without any a priori knowledge of their existence. Since the generalization of the scalars is fixed by the procedure, the only required input is the gauge transformation which we require the models to be invariant under. For this, we require the spin-n gauge transformations that are adopted in the literature [181, 20]. These generalizations are of the form

$$A'_\mu = A_\mu + \partial_\mu \xi \tag{4.77}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.78}$$

$$\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a \lambda_{b\rho} + \partial_b \lambda_{a\rho} + \partial_\rho \lambda_{ab} \tag{4.79}$$

$$\phi'_{b\chi\kappa\tau} = \phi_{b\chi\kappa\tau} + \partial_b \lambda_{\chi\kappa\tau} + \partial_\chi \lambda_{b\kappa\tau} + \partial_\kappa \lambda_{b\chi\tau} + \partial_\tau \lambda_{b\chi\kappa} \tag{4.80}$$

where the potentials, ϕ, and the gauge parameters, λ, are completely symmetric in all indices; this generalization continues to all n. We now have everything we need to apply the procedure for the $N = M = n$ case.

Scalars built from contracted spin-n curvature tensors for $N = M = 3$

For the $N = M = 3$ case, the most general Lagrangian density is the sum of all possible unique scalars of the form $\partial\partial\partial\phi\partial\partial\partial\phi$. This set of unique scalars is obtained by considering all the possible summation patterns that could occur in a scalar of the form $\partial\partial\partial\phi\partial\partial\partial\phi$. First, recognize that each scalar is the contraction of two terms, $\partial\partial\partial\phi$ and $\partial\partial\partial\phi$. Let us call these terms A and B. We can then group the possible scalars into four categories: scalars in which A and B each have 6 free indices (i.e. all the summation occurs between A and B, not within either), scalars in which A and B each have 4 free indices (i.e. summation occurs between A and B as well as within each), scalars in which A and B each have 2 free indices, and scalars in which A and B each have no free indices. Now, within each category, we consider all the possible ways the indices can sum. It is possible that an index on a derivative in A sums with an index on another derivative in A, with an index on ϕ in A, with an index on a derivative in B, or with an index on ϕ in B. Since ϕ is symmetric, these are the only unique possibilities. Likewise, it is possible that an index on ϕ in A sums with another index on ϕ in A, with an index on a derivative in A, with an index on ϕ in B, or with an index on a derivative in B.

Using these possibilities, we form all possible combinations of index sums within the contraction such that the resulting A and B terms have the given number of free indices. By writing out a term for each possible summation pattern within each category, we obtain a comprehensive set of all the unique scalars of the form $\partial\partial\partial\phi\partial\partial\partial\phi$. This set leads to the most general Lagrangian density

$$
\begin{aligned}
\mathcal{L} = {} & C_1\partial_\chi\partial_\tau\partial_\kappa\phi_{\gamma m\nu}\partial^\chi\partial^\tau\partial^\kappa\phi^{\gamma m\nu} + C_2\partial_\chi\partial_\tau\partial_\kappa\phi_{\gamma m\nu}\partial^\chi\partial^\tau\partial^\gamma\phi^{\kappa m\nu} + C_3\partial_\chi\partial_\tau\partial_\kappa\phi_{\gamma m\nu}\partial^\chi\partial^m\partial^\nu\phi^{\kappa\tau\gamma} \\
& + C_4\partial_\chi\partial_\tau\partial_\kappa\phi_{\gamma m\nu}\partial^\gamma\partial^m\partial^\nu\phi^{\chi\tau\kappa} + C_5\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\gamma\partial^\kappa\partial^\tau\phi^\gamma_{m\nu} + C_6\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_m\partial_\nu\partial_\gamma\phi^{\gamma\kappa\tau} \\
& + C_7\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\kappa\partial_m\partial_\gamma\phi^{\gamma\tau}_\nu + C_8\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\gamma\partial^\gamma\partial^\kappa\phi^\tau_{m\nu} + C_9\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\gamma\partial^\gamma\partial_\nu\phi^{\kappa\tau}_m \\
& + C_{10}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\tau\partial^\kappa\partial_m\phi^\gamma_{\gamma\nu} + C_{11}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\tau\partial_m\partial_\nu\phi^{\gamma\kappa}_\gamma + C_{12}\partial_\chi\partial^\chi\partial_\kappa\phi^{\tau\gamma m}\partial_\nu\partial^\nu\partial^\kappa\phi_{\tau\gamma m} \\
& + C_{13}\partial_\chi\partial^\chi\partial_\kappa\phi^{\tau\gamma m}\partial_\nu\partial^\nu\partial_\gamma\phi^\kappa_{\tau m} + C_{14}\partial_\chi\partial^\chi\partial_\kappa\phi^{\tau\gamma m}\partial_\tau\partial_\gamma\partial_m\phi^{\nu\kappa}_\nu + C_{15}\partial_\chi\partial^\chi\partial_\kappa\phi^{\tau\gamma m}\partial^\kappa\partial_\gamma\partial_m\phi^\nu_{\nu\tau} \\
& + C_{16}\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial^\chi\partial^\kappa\partial^\tau\phi^m_{m\nu} + C_{17}\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial_\nu\partial^\chi\partial^\kappa\phi^{m\tau}_m + C_{18}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau}_\gamma\partial_m\partial_\nu\partial^\kappa\phi^{m\nu\gamma} \\
& + C_{19}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\gamma}\partial_m\partial_\nu\partial_\gamma\phi^{m\nu\kappa} + C_{20}\partial_\chi\partial^\chi\partial_\tau\phi^{\tau\kappa}_\gamma\partial_\nu\partial^\nu\partial_m\phi^{m\gamma}_\kappa + C_{21}\partial_\chi\partial^\chi\partial_\tau\phi^{\tau\kappa\gamma}\partial_\kappa\partial_\gamma\partial_\nu\phi^{m\nu}_m \\
& + C_{22}\partial_\chi\partial^\chi\partial_\tau\phi^{\tau\kappa\gamma}\partial_\kappa\partial_m\partial_\nu\phi^{m\nu}_\gamma + C_{23}\partial_\chi\partial^\chi\partial_\tau\phi^{\tau\kappa\gamma}\partial_\kappa\partial_\nu\partial^\nu\phi^m_{m\gamma} + C_{24}\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial^\tau\partial_\nu\phi^{m\nu}_m \\
& + C_{25}\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial_m\partial_\nu\phi^{m\nu\tau} + C_{26}\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial_\nu\partial^\nu\partial^\chi\phi^{m\tau}_m + C_{27}\partial_\chi\partial^\chi\partial_\tau\phi^{\gamma\kappa}_\gamma\partial^\tau\partial_m\partial_\nu\phi^{m\nu}_\kappa \\
& + C_{28}\partial_\chi\partial^\chi\partial_\tau\phi^{\gamma\kappa}_\gamma\partial_\kappa\partial_m\partial_\nu\phi^{m\nu\tau} + C_{29}\partial_\chi\partial^\chi\partial_\tau\phi^{\gamma\kappa}_\gamma\partial_\nu\partial^\nu\partial^\tau\phi^m_{m\kappa} + C_{30}\partial_\chi\partial^\chi\partial_\tau\phi^{\gamma\kappa}_\gamma\partial_\nu\partial^\nu\partial_\kappa\phi^{m\tau}_m \\
& + C_{31}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\partial_m\partial_\nu\partial_\gamma\phi^{m\nu\gamma} + C_{32}\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\Box\partial^\nu\phi^\gamma_{\gamma\nu} + C_{33}\Box\partial_\tau\phi^{\chi\tau}_\chi\Box\partial^\nu\phi^\gamma_{\gamma\nu}, \quad (4.81)
\end{aligned}
$$

where $\Box = \partial_\alpha \partial^\alpha$. Note that, in the scalars multiplied by constants C_1 through C_4, A and B have 6 free indices, in the scalars multiplied by constants C_5 through C_{17}, A and B have 4 free indices, in the scalars multiplied by constants C_{18} through C_{30}, A and B have 2 free indices, and in the scalars multiplied by constants C_{31} through C_{33}, A and B have 0 free indices. This sorting is intentional: we will see that, as in the case of $N = M = 2$, these linear systems will decouple into factored curvature tensors of the 'Riemann' and 'Ricci' types. For clarity, we will treat these 4 types separately, as the $\mathcal{L}_6$, $\mathcal{L}_4$, $\mathcal{L}_2$ and $\mathcal{L}_0$ parts, respectively, of the general scalar, where the subscript refers to the number of free indices on A and B in the scalars of the given part. We can do this because the linear system of scalars identically decouples, with independent solutions for each of these four parts (there is no mixing between these four types of terms in the linear system of equations). Thus, for the above expression, we have $\mathcal{L} = \mathcal{L}_6 + \mathcal{L}_4 + \mathcal{L}_2 + \mathcal{L}_0$. Next we need to apply the gauge transformation $\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a \lambda_{b\rho} + \partial_b \lambda_{a\rho} + \partial_\rho \lambda_{ab}$ to the general scalar and solve for the free coefficients such that the remaining expression is exactly gauge invariant. **Solving the $\mathcal{L}_6$ system of linear equations for spin-3** Applying this

transformation to $\mathcal{L}_6$ and combining like terms yields

$$\begin{aligned}
\mathcal{L}_6 = {} & C_1 \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\chi \partial^\tau \partial^\kappa \phi^{\gamma m \nu} + C_2 \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\chi \partial^\tau \partial^\gamma \phi^{\kappa m \nu} + C_3 \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\chi \partial^m \partial^\nu \phi^{\kappa \tau \gamma} \\
& + C_4 \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\gamma \partial^m \partial^\nu \phi^{\chi \tau \kappa} + (6C_1 + 2C_2) \partial_\chi \partial_\tau \partial_\kappa \partial_\gamma \lambda_{m\nu} \partial^\chi \partial^\tau \partial^\kappa \phi^{\gamma m \nu} \\
& + (3C_1 + C_2) \partial_\chi \partial_\tau \partial_\kappa \partial_\gamma \lambda_{m\nu} \partial^\chi \partial^\tau \partial^\kappa \partial^\gamma \lambda^{m\nu} + (6C_1 + 6C_2 + 4C_3) \partial_\chi \partial_\tau \partial_\kappa \partial_\gamma \lambda_{m\nu} \partial^\chi \partial^\tau \partial^\kappa \partial^m \lambda^{\gamma\nu} \\
& + (4C_2 + 4C_3) \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\chi \partial^\tau \partial^\gamma \partial^m \lambda^{\kappa\nu} + (2C_2 + 5C_3 + 9C_4) \partial_\chi \partial_\tau \partial_\kappa \partial_m \lambda_{\gamma m} \partial^\chi \partial^\tau \partial^\gamma \partial^m \lambda^{\kappa\nu} \\
& + (2C_3 + 6C_4) \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} \partial^\chi \partial^m \partial^\nu \partial^\gamma \lambda^{\kappa\tau}. \quad (4.82)
\end{aligned}$$

The linear system of coefficients in front of the gauge parameter terms has the solution $C_1 = -C_4$, $C_2 = 3C_4$, $C_3 = -3C_4$ and $C_4 = C_4 = \tilde{A}$. Using this solution, the terms that depend on the gauge parameter all cancel and we are left with a gauge invariant expression. Remarkably, the remaining terms exactly factor into two independent 6 index tensors:

$$\begin{aligned}
\mathcal{L}_6 = \tilde{A} (& \partial_\chi \partial_\tau \partial_\kappa \phi_{\gamma m \nu} + \partial_\gamma \partial_\tau \partial_m \phi_{\chi \kappa \nu} + \partial_\gamma \partial_\nu \partial_\kappa \phi_{\chi m \tau} + \partial_\chi \partial_\nu \partial_m \phi_{\gamma \kappa \tau} \\
& - \partial_\gamma \partial_\nu \partial_m \phi_{\chi \kappa \tau} - \partial_\chi \partial_\tau \partial_m \phi_{\gamma \kappa \nu} - \partial_\chi \partial_\nu \partial_\kappa \phi_{\gamma m \tau} - \partial_\gamma \partial_\tau \partial_\kappa \phi_{\chi m \nu}) \\
\times (& \partial^\chi \partial^\tau \partial^\kappa \phi^{\gamma m \nu} + \partial^\tau \partial^m \partial^\gamma \phi^{\kappa \chi \nu} + \partial^\kappa \partial^\nu \partial^\gamma \phi^{\chi \tau m} + \partial^\chi \partial^m \partial^\nu \phi^{\kappa \tau \gamma} \\
& - \partial^\gamma \partial^m \partial^\nu \phi^{\chi \tau \kappa} - \partial^\chi \partial^\tau \partial^m \phi^{\kappa \nu \gamma} - \partial^\kappa \partial^\chi \partial^\nu \phi^{\tau m \gamma} - \partial^\tau \partial^\kappa \partial^\gamma \phi^{\chi m \nu}). \quad (4.83)
\end{aligned}$$

This is exactly the contraction of the spin-3 'Riemann' curvature tensor, $K^{\tau\nu\kappa m\chi\gamma}$, in equation (4.100)! Therefore we have derived the spin-3 'Riemann' curvature tensors by direct calculation. The contribution $\mathcal{L}_6$ has a unique gauge invariant solution, which is the contraction of the spin-3 curvature tensor: $\mathcal{L}_6 = C_4 K_{\tau\nu\kappa m\chi\gamma} K^{\tau\nu\kappa m\chi\gamma}$.

Solving the $\mathcal{L}_4$ system of linear equations for spin-3

Applying $\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a\lambda_{b\rho} + \partial_b\lambda_{a\rho} + \partial_\rho\lambda_{ab}$ to $\mathcal{L}_4$ and combining like terms yields

$$
\begin{aligned}
\mathcal{L}_4 = {}& C_5(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\gamma\partial^\kappa\partial^\tau\phi^\gamma_{m\nu}) + C_6(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_m\partial_\nu\partial_\gamma\phi^{\gamma\kappa\tau}) + C_7(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\kappa\partial_m\partial_\gamma\phi^{\gamma\tau}_\nu) \\
&+ C_8(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\kappa\Box\phi^\tau_{m\nu}) + C_9(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\nu\Box\phi^{\kappa\tau}_m) + C_{10}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\tau\partial^\kappa\partial_m\phi^\gamma_{\gamma\nu}) \\
&+ C_{11}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\tau\partial_m\partial_\nu\phi^{\gamma\kappa}_\gamma) + C_{12}(\partial_\kappa\Box\phi^{\tau\gamma m}\partial^\kappa\Box\phi_{\tau\gamma m}) + C_{13}(\partial_\kappa\Box\phi^{\tau\gamma m}\partial_\gamma\Box\phi^\kappa_{\tau m}) \\
&+ C_{14}(\partial_\kappa\Box\phi^{\tau\gamma m}\partial_\tau\partial_\gamma\partial_m\phi^{\nu\kappa}_\nu) + C_{15}(\partial_\kappa\Box\phi^{\tau\gamma m}\partial^\kappa\partial_\gamma\partial_m\phi^\nu_{\nu\tau}) + C_{16}(\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial^\chi\partial^\kappa\partial^\tau\phi^m_{m\nu}) \\
&+ C_{17}(\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial_\nu\partial^\chi\partial^\kappa\phi^{m\tau}_m) + (2C_5 + C_8)(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\kappa\partial^\tau\Box\lambda_{m\nu}) \\
&+ (4C_5 + 2C_7 + 2C_{10})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_\gamma\partial^\kappa\partial^\tau\partial_m\lambda^\gamma_\nu) + (2C_6 + C_9)(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_m\partial_\nu\Box\lambda^{\kappa\tau}) \\
&+ (4C_6 + 2C_7 + 2C_{11})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial_m\partial_\nu\partial_\gamma\partial^\kappa\lambda^{\gamma\tau}) + (2C_7 + 2C_8 + 2C_9)(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\kappa\partial_m\Box\lambda^\tau_\nu) \\
&+ (C_8 + 6C_{12} + 2C_{13})(\partial_\kappa\Box\phi^{\tau\gamma m}\partial^\kappa\partial_\tau\Box\lambda_{\gamma m}) + (2C_8 + C_9 + 2C_{15})(\partial_\kappa\Box\phi^{\tau\gamma m}\partial^\kappa\partial_\gamma\partial_m\partial^\nu\lambda_{\tau\nu}) \\
&+ (C_9 + 4C_{13})(\partial_\kappa\Box\phi^{\tau\gamma m}\partial_\gamma\partial_\tau\Box\lambda^\kappa_m) + (C_9 + 2C_{14})(\partial_\kappa\Box\phi^{\tau\gamma m}\partial_\tau\partial_\gamma\partial_m\partial^\nu\lambda^\kappa_\nu) \\
&+ (C_{10} + C_{11})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi m\nu}\partial^\tau\partial_m\partial_\nu\partial^\kappa\lambda^\gamma_\gamma) + (C_{10} + 2C_{15})(\partial_\kappa\partial^\gamma\Box\lambda^{\tau m}\partial^\kappa\partial_\gamma\partial_m\phi^\nu_{\nu\tau}) \\
&+ (C_{10} + 2C_{11} + 4C_{17})(\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial_\nu\partial^\chi\partial^\kappa\partial^m\lambda^\tau_m) + (C_{10} + 4C_{16})(\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial^\chi\partial^\kappa\partial^\tau\partial^m\lambda_{\nu m}) \\
&+ (C_{11} + 3C_{14} + C_{15})(\partial_\kappa\partial^\tau\Box\lambda^{\gamma m}\partial_\tau\partial_\gamma\partial_m\phi^{\nu\kappa}_\nu) + (C_{14} + C_{15})(\partial_\kappa\Box\phi^{\tau\gamma m}\partial_\tau\partial_\gamma\partial_m\partial^\kappa\lambda^\nu_\nu) \\
&+ (2C_{16} + 2C_{17})(\partial_\chi\partial_\kappa\partial_\tau\phi^{\gamma\nu}_\gamma\partial_\nu\partial^\chi\partial^\kappa\partial^\tau\lambda^m_m) + (C_5 + C_8 + 3C_{12} + C_{13})(\partial_\kappa\partial^\gamma\Box\lambda^{\tau m}\partial_\gamma\partial^\kappa\Box\lambda_{\tau m}) \\
&+ (4C_5 + 2C_7 + 4C_8 + 2C_9 + 2C_{10} + 4C_{15})(\partial_\kappa\partial^\gamma\Box\lambda^{\tau m}\partial^\kappa\partial_\gamma\partial_m\partial^\nu\lambda_{\tau\nu}) \\
&+ (2C_5 + C_7 + 2C_{10} + 4C_{16})(\partial_\chi\partial_\kappa\partial_\tau\partial^\gamma\lambda^\nu_\gamma\partial^\chi\partial^\kappa\partial^\tau\partial^m\lambda_{\nu m}) \\
&+ (2C_5 + 4C_6 + 3C_7 + 2C_{10} + 4C_{11} + 4C_{17})(\partial_\chi\partial_\kappa\partial_\tau\partial^\gamma\lambda^\nu_\gamma\partial_\nu\partial^\chi\partial^\kappa\partial^m\lambda^\tau_m) \\
&+ (C_6 + C_9 + 2C_{13})(\partial_\tau\partial_\kappa\Box\lambda^{m\nu}\partial_m\partial_\nu\Box\lambda^{\kappa\tau}) + (C_{10} + C_{11} + 3C_{14} + 3C_{15})(\partial_\kappa\partial^\tau\Box\lambda^{\gamma m}\partial_\tau\partial_\gamma\partial_m\partial^\kappa\lambda^\nu_\nu) \\
&+ (4C_6 + 2C_7 + 2C_8 + 4C_9 + 2C_{11} + 6C_{14} + 2C_{15})(\partial_\kappa\partial^\tau\Box\lambda^{\gamma m}\partial^\kappa\partial_\gamma\partial_m\partial^\nu\lambda_{\tau\nu}) \\
&+ (C_7 + 2C_8 + 2C_9 + 6C_{12} + 6C_{13})(\partial_\kappa\partial^\tau\Box\lambda^{\gamma m}\partial_\gamma\partial^\kappa\Box\lambda_{\tau m}) \\
&+ (2C_{10} + 2C_{11} + 4C_{16} + 4C_{17})(\partial_\chi\partial_\tau\partial_\kappa\partial^m\lambda^{\chi\nu}\partial^\tau\partial_m\partial_\nu\partial^\kappa\lambda^\gamma_\gamma) + (C_{16} + C_{17})(\partial_\chi\partial_\kappa\partial_\tau\partial^\nu\lambda^\gamma_\gamma\partial_\nu\partial^\chi\partial^\kappa\partial^\tau\lambda^m_m),
\end{aligned}
\tag{4.84}
$$

which has the solution $C_5 = -2C_{17}$, $C_6 = 2C_{17}$, $C_7 = 0$, $C_8 = 4C_{17}$, $C_9 = -4C_{17}$, $C_{10} = 4C_{17}$, $C_{11} = -4C_{17}$, $C_{12} = -C_{17}$, $C_{13} = C_{17}$, $C_{14} = 2C_{17}$, $C_{15} = -2C_{17}$, $C_{16} = -C_{17}$ and $C_{17} = C_{17} = \tilde{B}$. Using this solution, the terms that depend on the gauge parameter all cancel and we are left with a gauge invariant expression. Remarkably, the remaining terms exactly factor into two independent 4 index tensors:

$$\begin{aligned}\mathcal{L}_4 = \tilde{B}(&\partial^\nu\Box\phi^{\tau\kappa m} + \partial^m\partial^\nu\partial^\tau\phi_\gamma^{\gamma\kappa} + \partial_\chi\partial^\tau\partial^\kappa\phi^{\chi m\nu} + \partial_\chi\partial^m\partial^\kappa\phi^{\chi\tau\nu} \\ &- \partial^\kappa\Box\phi^{\tau\nu m} - \partial^m\partial^\kappa\partial^\tau\phi_\gamma^{\gamma\nu} - \partial_\chi\partial^\tau\partial^\nu\phi^{\chi m\kappa} - \partial_\chi\partial^m\partial^\nu\phi^{\chi\tau\kappa}) \\ &\times(\partial_\nu\Box\phi_{\tau\kappa m} + \partial_\nu\partial_\tau\partial_m\phi^\gamma_{\gamma\kappa} + \partial_\tau\partial_\kappa\partial^\chi\phi_{\chi m\nu} + \partial_m\partial_\kappa\partial^\chi\phi_{\chi\tau\nu} \\ &- \partial_\kappa\Box\phi_{\nu\tau m} - \partial_\tau\partial_\kappa\partial_m\phi^\gamma_{\gamma\nu} - \partial_m\partial_\nu\partial^\chi\phi_{\chi\tau\kappa} - \partial_\tau\partial_\nu\partial^\chi\phi_{\chi m\kappa}).\end{aligned} \tag{4.85}$$

But this is exactly the contraction of the first spin-3 ‘Ricci’ curvature tensor, $K^{\nu\kappa\tau m}$! Therefore, $\mathcal{L}_4$ has a unique gauge invariant solution, which is the contraction of the first spin-3 ‘Ricci’ curvature tensor: $\mathcal{L}_4 = \tilde{B}4K_{\nu\kappa\tau m}K^{\nu\kappa\tau m}$. **Solving the $\mathcal{L}_2$ system of linear equations for spin-3**

Applying $\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a\lambda_{b\rho} + \partial_b\lambda_{a\rho} + \partial_\rho\lambda_{ab}$ to $\mathcal{L}_2$ and combining like terms yields:

$$
\begin{aligned}
\mathcal{L}_2 = {} & C_{18}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau}_\gamma\partial_m\partial_\nu\partial^\kappa\phi^{m\nu\gamma}) + C_{19}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\gamma}\partial_m\partial_\nu\partial_\gamma\phi^{m\nu\kappa}) + C_{20}(\partial_\tau\Box\phi^{\tau\kappa}_\gamma\partial_m\Box\phi^{m\gamma}_\kappa) \\
& + C_{21}(\partial_\tau\Box\phi^{\tau\kappa\gamma}\partial_\kappa\partial_\gamma\partial_\nu\phi^{m\nu}_m) + C_{22}(\partial_\tau\Box\phi^{\tau\kappa\gamma}\partial_\kappa\partial_m\partial_\nu\phi^{m\nu}_\gamma) + C_{23}(\partial_\tau\Box\phi^{\tau\kappa\gamma}\partial_\kappa\Box\phi^m_{m\gamma}) \\
& + C_{24}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial^\tau\partial_\nu\phi^{m\nu}_m) + C_{25}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial_m\partial_\nu\phi^{m\nu\tau}) + C_{26}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\Box\partial^\chi\phi^{m\tau}_m) \\
& + C_{27}(\Box\partial_\tau\phi^{\gamma\kappa}_\gamma\partial^\tau\partial_m\partial_\nu\phi^{m\nu}_\kappa) + C_{28}(\Box\partial_\tau\phi^{\gamma\kappa}_\gamma\partial_\kappa\partial_m\partial_\nu\phi^{m\nu\tau}) + C_{29}(\partial_\tau\Box\phi^{\gamma\kappa}_\gamma\partial^\tau\Box\phi^m_{m\kappa}) \\
& + C_{30}(\partial_\tau\Box\phi^{\gamma\kappa}_\gamma\partial_\kappa\Box\phi^{m\tau}_m) + (4C_{18} + C_{22} + 2C_{27})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau}_\gamma\partial_\nu\partial^\kappa\Box\lambda^{\nu\gamma}) \\
& + (2C_{18} + 2C_{19} + 2C_{25})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\gamma}\partial_m\partial_\nu\partial_\gamma\partial^\kappa\lambda^{m\nu}) \\
& + (4C_{19} + C_{22} + 2C_{28})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\gamma}\partial_\nu\partial_\gamma\Box\lambda^{\nu\kappa}) + 2C_{20}(\partial_\tau\Box\phi^{\tau\kappa}_\gamma\Box\Box\lambda^\gamma_\kappa) \\
& + (4C_{20} + 2C_{22} + 2C_{23})(\partial_\tau\Box\phi^{\tau\kappa}_\gamma\partial_m\partial^\gamma\Box\lambda^m_\kappa) + (2C_{21} + C_{22})(\partial_\tau\Box\phi^{\tau\kappa\gamma}\partial_\kappa\partial_\gamma\partial_\nu\partial^m\lambda^\nu_m) \\
& + (C_{21} + C_{23})(\partial_\tau\Box\phi^{\tau\kappa\gamma}\partial_\kappa\partial_\gamma\Box\lambda^m_m) + (2C_{21} + 2C_{25} + 2C_{26})(\partial_\tau\partial^\kappa\Box\lambda^{\tau\gamma}\partial_\kappa\partial_\gamma\partial_\nu\phi^{m\nu}_m) \\
& + C_{21}(\Box\Box\lambda^{\kappa\gamma}\partial_\kappa\partial_\gamma\partial_\nu\phi^{m\nu}_m) + C_{22}(\Box\Box\lambda^{\kappa\gamma}\partial_\kappa\partial_m\partial_\nu\phi^{m\nu}_\gamma) + (C_{23} + 2C_{27} + 4C_{29})(\partial_\tau\partial^\kappa\Box\lambda^{\tau\gamma}\partial_\kappa\Box\phi^m_{m\gamma}) \\
& + C_{23}(\Box\Box\lambda^{\kappa\gamma}\partial_\kappa\Box\phi^m_{m\gamma}) + (C_{23} + 2C_{28} + 4C_{30})(\partial_\tau\Box\phi^{\gamma\kappa}_\gamma\partial_\kappa\partial^m\Box\lambda^\tau_m) \\
& + (4C_{24} + C_{25})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial^\tau\partial_\nu\partial^m\lambda^\nu_m) + (C_{25} + C_{27} + C_{28})(\partial_\tau\partial^\kappa\Box\lambda^\gamma_\gamma\partial_\kappa\partial_m\partial_\nu\phi^{m\nu\tau}) \\
& + (2C_{24} + C_{26})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\gamma\kappa}_\gamma\partial^\chi\partial^\tau\Box\lambda^m_m) + (2C_{26} + C_{27} + C_{28})(\Box\partial_\tau\phi^{\gamma\kappa}_\gamma\partial_\kappa\partial_m\partial_\nu\partial^\tau\lambda^{m\nu}) \\
& + (C_{26} + 2C_{29} + 2C_{30})(\partial_\tau\Box\phi^{\gamma\kappa}_\gamma\partial_\kappa\partial^\tau\Box\lambda^m_m) + (C_{24} + C_{26} + C_{29} + C_{30})(\partial_\tau\partial^\kappa\Box\lambda^\gamma_\gamma\partial_\kappa\partial^\tau\Box\lambda^m_m) \\
& + (4C_{18} + 2C_{20} + 2C_{22} + 2C_{23} + 4C_{27} + 4C_{29})(\partial_\tau\partial_\kappa\Box\lambda^\tau_\gamma\partial_\nu\partial^\kappa\Box\lambda^{\nu\gamma}) \\
& + (4C_{18} + 4C_{19} + 4C_{21} + 2C_{22} + 4C_{25} + 4C_{26} + 2C_{27} + 2C_{28})(\partial_\tau\partial_\kappa\Box\lambda^\tau_\gamma\partial_m\partial_\nu\partial^\kappa\partial^\gamma\lambda^{m\nu}) \\
& + (C_{18} + C_{19} + 4C_{24} + 2C_{25})(\partial_\chi\partial_\tau\partial_\kappa\partial^\gamma\lambda^\kappa_\gamma\partial^\chi\partial_m\partial_\nu\partial^\tau\lambda^{m\nu}) \\
& + (4C_{19} + 2C_{20} + 2C_{22} + 2C_{23} + 4C_{28} + 4C_{30})(\partial_\tau\partial^\gamma\Box\lambda^\kappa_\gamma\partial_\kappa\partial^m\Box\lambda^\tau_m) \\
& + (4C_{20} + 2C_{22} + 2C_{23})(\partial_\tau\partial^\kappa\Box\lambda^\tau_\gamma\Box\Box\lambda^\gamma_\kappa) + C_{20}(\Box\Box\lambda^\kappa_\gamma\Box\Box\lambda^\gamma_\kappa) \\
& + (2C_{21} + 2C_{23} + 2C_{25} + 2C_{26} + 2C_{27} + 2C_{28} + 4C_{29} + 4C_{30})(\partial_\tau\partial^\gamma\Box\lambda^\kappa_\gamma\partial_\kappa\partial^\tau\Box\lambda^m_m) \\
& + (2C_{21} + C_{22})(\Box\Box\lambda^{\kappa\gamma}\partial_\kappa\partial_\gamma\partial_\nu\partial^m\lambda^\nu_m) + (C_{21} + C_{23})(\Box\Box\lambda^{\kappa\gamma}\partial_\kappa\partial_\gamma\Box\lambda^m_m) \\
& + (4C_{24} + C_{25} + 2C_{26} + C_{27} + C_{28})(\partial_\tau\partial^\kappa\Box\lambda^\gamma_\gamma\partial_\kappa\partial_m\partial_\nu\partial^\tau\lambda^{m\nu}),
\end{aligned}
\tag{4.86}
$$

which has the solution $C_{18} = C_{18} = \tilde{C}$, $C_{19} = -\tilde{C}$, $C_{20} = 0$, $C_{21} = 0$, $C_{22} = 0$, $C_{23} = 0$, $C_{24} = 0$, $C_{25} = 0$, $C_{26} = 0$, $C_{27} = -2\tilde{C}$, $C_{28} = 2\tilde{C}$, $C_{29} = \tilde{C}$, and $C_{30} = -\tilde{C}$. Using this solution, the terms that depend on the gauge parameter all cancel and we are left with a gauge invariant expression. Remarkably, the remaining terms exactly factor into two independent 2 index tensors:

$$
\begin{aligned}
\mathcal{L}_2 = {} & \tilde{C}(\partial_m\partial_\nu\partial^\kappa\phi^{m\nu\gamma} - \partial_m\partial_\nu\partial^\gamma\phi^{m\nu\kappa} + \partial^\gamma\Box\phi^{m\kappa}_m - \partial^\kappa\Box\phi^{m\gamma}_m) \\
& \times (\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau}_\gamma - \partial_\chi\partial_\tau\partial_\gamma\phi^{\chi\tau}_\kappa - \Box\partial_\kappa\phi^\chi_{\chi\gamma} + \Box\partial_\gamma\phi^\chi_{\chi\kappa}).
\end{aligned}
\tag{4.87}
$$

But this is exactly the contraction of the second spin-3 'Ricci' curvature tensor, $K^{\kappa\gamma}$! Therefore, $\mathcal{L}_2$ has a unique gauge invariant solution, which is the contraction of the second spin-3 'Ricci' curvature tensor: $\mathcal{L}_2 = \tilde{C}K^{\kappa\gamma}K_{\kappa\gamma}$.

Solving the $\mathcal{L}_0$ system of linear equations for spin-3

Applying $\phi'_{ab\rho} = \phi_{ab\rho} + \partial_a\lambda_{b\rho} + \partial_b\lambda_{a\rho} + \partial_\rho\lambda_{ab}$ to $\mathcal{L}_0$ and combining like terms yields

$$\begin{aligned}\mathcal{L}_0 = C_{31}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\partial_m\partial_\nu\partial_\gamma\phi^{m\nu\gamma}) + C_{32}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\Box\partial^\nu\phi^\gamma_{\gamma\nu}) \\ + C_{33}(\Box\partial_\tau\phi^{\chi\tau}_\chi\Box\partial^\nu\phi^\gamma_{\gamma\nu}) + (6C_{31} + 2C_{32})(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\partial_m\partial_\nu\Box\lambda^{\nu m}) \\ + C_{32}(\partial_\chi\partial_\tau\partial_\kappa\phi^{\chi\tau\kappa}\Box\Box\lambda^\gamma_\gamma) + (3C_{32} + 4C_{33})(\Box\partial_\tau\phi^{\chi\tau}_\chi\Box\partial^\nu\partial^\gamma\lambda_{\nu\gamma}) \\ + 2C_{33}(\Box\partial_\tau\phi^{\chi\tau}_\chi\Box\Box\lambda^\gamma_\gamma) + (9C_{31} + 6C_{32} + 4C_{33})(\partial_\tau\partial_\kappa\Box\lambda^{\tau\kappa}\partial_m\partial_\nu\Box\lambda^{\nu m}) \\ + (3C_{32} + 4C_{33})(\Box\partial_\tau\partial^\chi\lambda^\tau_\chi\Box\Box\lambda^\gamma_\gamma) + C_{33}(\Box\Box\lambda^\chi_\chi\Box\Box\lambda^\gamma_\gamma),\end{aligned} \quad (4.88)$$

which has the solution $C_{31} = C_{32} = C_{33} = 0$. Thus, $\mathcal{L}_0 = 0$. This result is easily understood, since spin-n models for $n = odd$ will have a scalar curvature tensor equal to zero, as in the case of electrodynamics, where $F = \eta_{\mu\nu}F^{\mu\nu} = 0$. Therefore, combining all parts, the Lagrangian density for the spin-3 case is

$$\mathcal{L} = \tilde{A}K^{\tau\nu\kappa m\chi\gamma}K_{\tau\nu\kappa m\chi\gamma} + \tilde{B}K^{\tau\nu\kappa\chi}K_{\tau\nu\kappa\chi} + \tilde{C}K^{\nu\chi}K_{\nu\chi}. \quad (4.89)$$

Scalars built from contracted spin-n curvature tensors for $N = M = 4$

For the $N = M = 4$ case, the most general Lagrangian density is the sum of all possible unique scalars of the form $\partial\partial\partial\partial\phi\partial\partial\partial\partial\phi$. This set of unique scalars is obtained by considering all the possible summation patterns that could occur in a scalar of the form $\partial\partial\partial\partial\phi\partial\partial\partial\partial\phi$. This is done using exactly the same method as for the $N = M = 3$ case, except that, for $N = M = 4$, there are more possible summation patterns since there are 4 derivatives and 4 indices on ϕ. Again we consider each scalar as a contraction of two terms which we call A and B and we group the possible scalars into categories based on the free indices of A and B. In this case, it is possible for A and B to have 8, 6, 4, 2 or 0 free indices, so we have 5 categories. Next, as before, within each category, we consider all the possible contractions. By writing out a term for each possible summation pattern within each category, we obtain a comprehensive set of all the unique scalars of the form $\partial\partial\partial\partial\phi\partial\partial\partial\partial\phi$.

Again, the system of linear equations which is solved to find the values of the constant coefficients of $\mathcal{L}$ will decouple into independent linear systems based on the number of free indices (8, 6, 4, 2 or 0) on the A and B terms in the scalars the coefficients multiply. Therefore, we will again treat the most general Lagrangian density as the sum of the five types of Lagrangian densities; the most general Lagrangian density for the spin-4 case will then be of the form $\mathcal{L} = \mathcal{L}_8 + \mathcal{L}_6 + \mathcal{L}_4 + \mathcal{L}_2 + \mathcal{L}_0$. For brevity (and because performing this calculation entirely by hand is a bit crazy), we will only directly solve for $\mathcal{L}_8$, since we know from the $N = M = 2$ and $N = M = 3$ cases that the remaining terms are built from contractions of the 'Riemann' curvature tensor. The most general representation of $\mathcal{L}_8$ is

$$\begin{aligned}\mathcal{L}_8 = D_1\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\nu\phi^{b\chi\kappa\tau} + D_2\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\tau\phi^{b\chi\kappa\nu}\\ + D_3\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^\kappa\partial^\tau\phi^{b\chi m\nu} + D_4\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\chi\partial^\kappa\partial^\tau\phi^{b\gamma m\nu}\\ + D_5\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^b\partial^\chi\partial^\kappa\partial^\tau\phi^{a\gamma m\nu}.\end{aligned} \tag{4.90}$$

Next, we need to apply the gauge transformation $\phi'_{b\chi\kappa\tau} = \phi_{b\chi\kappa\tau} + \partial_b\lambda_{\chi\kappa\tau} + \partial_\chi\lambda_{b\kappa\tau} + \partial_\kappa\lambda_{b\chi\tau} + \partial_\tau\lambda_{b\chi\kappa}$ to the general scalar and solve for the free coefficients such that the remaining expression is exactly gauge invariant. Applying this transformation to $\mathcal{L}_8$ and combining like terms yields

$$\begin{aligned}\mathcal{L}_8 = D_1(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\nu\phi^{b\chi\kappa\tau}) + D_2(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\tau\phi^{b\chi\kappa\nu})\\ + D_3(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^\kappa\partial^\tau\phi^{b\chi m\nu}) + D_4(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\chi\partial^\kappa\partial^\tau\phi^{b\gamma m\nu})\\ + D_5(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^b\partial^\chi\partial^\kappa\partial^\tau\phi^{a\gamma m\nu}) + (8D_1 + 2D_2)(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\nu\partial^b\lambda^{\chi\kappa\tau})\\ + (8D_5 + 2D_4)(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^b\partial^\chi\partial^\kappa\partial^\tau\partial^a\lambda^{\gamma m\nu}) + (6D_4 + 4D_3)(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^\kappa\partial^\tau\partial^b\lambda^{\chi m\nu})\\ +(4D_3+6D_2)(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\tau\partial^b\lambda^{\chi\kappa\nu})+(2D_3+16D_5+7D_4)(\partial_a\partial_\gamma\partial_m\partial_\nu\partial_b\lambda_{\chi\kappa\tau}\partial^a\partial^\gamma\partial^\kappa\partial^\tau\partial^\chi\lambda^{bm\nu})\\ + (9D_4 + 10D_3 + 6D_2)(\partial_a\partial_\gamma\partial_m\partial_\nu\partial_\chi\lambda_{b\kappa\tau}\partial^a\partial^\chi\partial^\kappa\partial^\tau\partial^\gamma\lambda^{bm\nu})\\ +(12D_1+9D_2+4D_3)(\partial_a\partial_\gamma\partial_m\partial_\nu\partial_b\lambda_{\chi\kappa\tau}\partial^a\partial^\gamma\partial^m\partial^\nu\partial^\chi\lambda^{b\kappa\tau})+(4D_1+D_2)(\partial_a\partial_\gamma\partial_m\partial_\nu\partial_\tau\lambda_{b\chi\kappa}\partial^a\partial^\gamma\partial^m\partial^\tau\partial^\nu\lambda^{b\chi\kappa}).\end{aligned} \tag{4.91}$$

The linear system of coefficients has the solution $D_1 = D_1 = \tilde{D}$, $D_2 = -4\tilde{D}$, $D_3 = 6\tilde{D}$, $D_4 = -4\tilde{D}$ and $D_5 = \tilde{D}$. Using this solution, the terms that depend on the gauge parameter all cancel and we are left with a gauge invariant expression. Remarkably, the remaining terms exactly factor into two independent 8 index tensors:

$$
\begin{aligned}
\mathcal{L}_8 = \tilde{D}\Big(&\partial^a\partial^\gamma\partial^m\partial^\nu\phi^{b\chi\kappa\tau} + \partial^a\partial^\gamma\partial^\kappa\partial^\tau\phi^{b\chi m\nu} + \partial^a\partial^m\partial^\chi\partial^\tau\phi^{b\kappa\gamma\nu} + \partial^a\partial^\nu\partial^\chi\partial^\kappa\phi^{b\tau\gamma m} \\
&+ \partial^\gamma\partial^m\partial^b\partial^\tau\phi^{\chi\kappa a\nu} + \partial^\gamma\partial^\nu\partial^b\partial^\kappa\phi^{\chi\tau a m} + \partial^m\partial^\nu\partial^b\partial^\chi\phi^{\kappa\tau a\gamma} + \partial^b\partial^\chi\partial^\kappa\partial^\tau\phi^{a\gamma m\nu} \\
&- \partial^a\partial^\gamma\partial^m\partial^\tau\phi^{b\chi\kappa\nu} - \partial^a\partial^\gamma\partial^\nu\partial^\kappa\phi^{b\chi\tau m} - \partial^a\partial^m\partial^\nu\partial^\chi\phi^{b\kappa\tau\gamma} - \partial^\gamma\partial^m\partial^\nu\partial^b\phi^{\chi\kappa\tau a} \\
&- \partial^a\partial^\chi\partial^\kappa\partial^\tau\phi^{b\gamma m\nu} - \partial^\gamma\partial^b\partial^\kappa\partial^\tau\phi^{\chi a m\nu} - \partial^m\partial^b\partial^\chi\partial^\tau\phi^{\kappa a\gamma\nu} - \partial^\nu\partial^b\partial^\chi\partial^\kappa\phi^{\tau a\gamma m}\Big) \\
&\times\Big(\partial_a\partial_\gamma\partial_m\partial_\nu\phi_{b\chi\kappa\tau} + \partial_a\partial_\gamma\partial_\kappa\partial_\tau\phi_{b\chi m\nu} + \partial_a\partial_\chi\partial_m\partial_\tau\phi_{b\gamma\kappa\nu} + \partial_a\partial_\chi\partial_\kappa\partial_\nu\phi_{b\gamma m\tau} \\
&+ \partial_b\partial_\gamma\partial_m\partial_\tau\phi_{a\chi\kappa\nu} + \partial_b\partial_\gamma\partial_\kappa\partial_\nu\phi_{a\chi m\tau} + \partial_b\partial_\chi\partial_m\partial_\nu\phi_{a\gamma\kappa\tau} + \partial_b\partial_\chi\partial_\kappa\partial_\tau\phi_{a\gamma m\nu} \\
&- \partial_a\partial_\gamma\partial_m\partial_\tau\phi_{b\chi\kappa\nu} - \partial_a\partial_\gamma\partial_\kappa\partial_\nu\phi_{b\chi m\tau} - \partial_a\partial_\chi\partial_m\partial_\nu\phi_{b\gamma\kappa\tau} - \partial_b\partial_\gamma\partial_m\partial_\nu\phi_{a\chi\kappa\tau} \\
&\quad - \partial_a\partial_\chi\partial_\kappa\partial_\tau\phi_{b\gamma m\nu} - \partial_b\partial_\gamma\partial_\kappa\partial_\tau\phi_{a\chi m\nu} - \partial_b\partial_\chi\partial_m\partial_\tau\phi_{a\gamma\kappa\nu} - \partial_b\partial_\chi\partial_\kappa\partial_\nu\phi_{a\gamma m\tau}\Big). \qquad (4.92)
\end{aligned}
$$

But this is exactly the contraction of the spin-4 'Riemann' curvature tensor, $K^{ab\gamma\chi m\kappa\nu\tau}$, in equation (4.101)! Therefore we have derived the spin-4 'Riemann' curvature tensor by direct calculation. The contribution $\mathcal{L}_8$ has a unique gauge invariant solution, which is the contraction of the spin-4 'Riemann' curvature tensor: $\mathcal{L}_8 = \tilde{D}K^{ab\gamma\chi m\kappa\nu\tau}K_{ab\gamma\chi m\kappa\nu\tau}$.

In order to determine the possible form of the remaining spin-4 scalars, we will contract the curvature tensor in equation (4.101); these contractions are known so we include them only for completeness. First, to find the rank 6 'Ricci' curvature tensor, we contract $\eta_{ab}K^{a\gamma b\chi m\kappa\nu\tau}$ to yield

$$
\begin{aligned}
K^{\gamma\chi m\kappa\nu\tau} = &\,\partial^m\partial^\nu\Box\phi^{\gamma\chi\kappa\tau} + \partial^\kappa\partial^\tau\Box\phi^{\gamma\chi m\nu} + \partial_a\partial^m\partial^\chi\partial^\tau\phi^{\gamma\kappa a\nu} + \partial_a\partial^\nu\partial^\chi\partial^\kappa\phi^{\gamma\tau a m} \\
&+ \partial_a\partial^m\partial^\gamma\partial^\tau\phi^{\chi\kappa a\nu} + \partial_a\partial^\nu\partial^\gamma\partial^\kappa\phi^{\chi\tau a m} + \partial^m\partial^\nu\partial^\gamma\partial^\chi\phi_a^{a\kappa\tau} + \partial^\gamma\partial^\chi\partial^\kappa\partial^\tau\phi_a^{a m\nu} \\
&- \partial^m\partial^\tau\Box\phi^{\gamma\chi\kappa\nu} - \partial^\nu\partial^\kappa\Box\phi^{\gamma\chi\tau m} - \partial_a\partial^m\partial^\nu\partial^\chi\phi^{\gamma\kappa\tau a} - \partial_a\partial^m\partial^\nu\partial^\gamma\phi^{\chi\kappa\tau a} \\
&\quad - \partial_a\partial^\chi\partial^\kappa\partial^\tau\phi^{\gamma a m\nu} - \partial_a\partial^\gamma\partial^\kappa\partial^\tau\phi^{\chi a m\nu} - \partial^m\partial^\gamma\partial^\chi\partial^\tau\phi_a^{a\kappa\nu} - \partial^\nu\partial^\gamma\partial^\chi\partial^\kappa\phi_a^{a\tau m}. \qquad (4.93)
\end{aligned}
$$

This expression has the symmetries $K^{\gamma\chi m\kappa\nu\tau} = K^{\chi\gamma m\kappa\nu\tau} = K^{\gamma\chi\nu\tau m\kappa} = -K^{\gamma\chi\kappa m\nu\tau} = -K^{\gamma\chi m\kappa\tau\nu}$. There are three possible rank 4 contractions for the second 'Ricci' curvature, although two are redundant (not independent). The first (4.94) is found by contracting one of the indices in the symmetric pair and one in one of the antisymmetric pairs $\eta_{\kappa m}K^{m\chi\kappa\gamma\nu\tau}$. The second (4.95) is found by contracting one of the indices from each of the antisymmetric pairs $\eta_{m\kappa}K^{\chi\gamma m\nu\kappa\tau}$. The third that is equivalent to the first (4.94) is found by contracting both the indices of the symmetric pair $\eta_{\kappa m}K^{\kappa m\chi\gamma\nu\tau}$. The rank 4 tensors are:

$$K^{\chi\gamma\nu\tau} = 2\partial^{\chi}\partial^{\nu}\Box\phi^{\kappa\gamma\tau}_{\kappa} + 2\partial^{\gamma}\partial^{\tau}\Box\phi^{\kappa\chi\nu}_{\kappa} + 2\partial_{\kappa}\partial_{a}\partial^{\chi}\partial^{\tau}\phi^{\kappa\gamma a\nu} + 2\partial_{\kappa}\partial_{a}\partial^{\nu}\partial^{\gamma}\phi^{\kappa\tau a\chi}$$
$$- 2\partial^{\chi}\partial^{\tau}\Box\phi^{\kappa\gamma\nu}_{\kappa} - 2\partial^{\nu}\partial^{\gamma}\Box\phi^{\kappa\tau\chi}_{\kappa} - 2\partial_{\kappa}\partial_{a}\partial^{\chi}\partial^{\nu}\phi^{\kappa\gamma\tau a} - 2\partial_{\kappa}\partial_{a}\partial^{\gamma}\partial^{\tau}\phi^{\kappa a\chi\nu}, \quad (4.94)$$

$$\hat{K}^{\chi\gamma\nu\tau} = \Box\Box\phi^{\chi\gamma\nu\tau} + \partial^{\nu}\partial^{\tau}\Box\phi^{\kappa\chi\gamma}_{\kappa} + \partial_{\kappa}\partial_{a}\partial^{\gamma}\partial^{\tau}\phi^{\chi\nu a\kappa} + \partial_{a}\partial_{\kappa}\partial^{\gamma}\partial^{\nu}\phi^{\chi\tau a\kappa}$$
$$+ \partial_{a}\partial_{\kappa}\partial^{\chi}\partial^{\tau}\phi^{\gamma\nu a\kappa} + \partial_{a}\partial_{\kappa}\partial^{\chi}\partial^{\nu}\phi^{\gamma\tau a\kappa} + \partial^{\chi}\partial^{\gamma}\Box\phi^{a\nu\tau}_{a} + \partial^{\chi}\partial^{\gamma}\partial^{\nu}\partial^{\tau}\phi^{a\kappa}_{a\kappa}$$
$$- \partial_{\kappa}\partial^{\tau}\Box\phi^{\chi\gamma\nu\kappa} - \partial_{\kappa}\partial^{\nu}\Box\phi^{\chi\gamma\tau\kappa} - \partial_{a}\partial^{\gamma}\Box\phi^{\chi\nu\tau a} - \partial_{a}\partial^{\chi}\Box\phi^{\gamma\nu\tau a}$$
$$- \partial_{a}\partial^{\gamma}\partial^{\nu}\partial^{\tau}\phi^{\kappa\chi a}_{\kappa} - \partial_{a}\partial^{\chi}\partial^{\nu}\partial^{\tau}\phi^{\kappa\gamma a}_{\kappa} - \partial_{\kappa}\partial^{\chi}\partial^{\gamma}\partial^{\tau}\phi^{a\nu\kappa}_{a} - \partial_{\kappa}\partial^{\chi}\partial^{\gamma}\partial^{\nu}\phi^{a\tau\kappa}_{a}, \quad (4.95)$$

where $K^{\chi\gamma\nu\tau}$ in (4.94) has symmetries and anti-symmetries $\bar{K}^{\chi\gamma\nu\tau} = K^{\nu\tau\chi\gamma} = -K^{\gamma\chi\nu\tau} = -K^{\chi\gamma\tau\nu}$ and $\hat{K}^{\chi\gamma\nu\tau}$ in (4.95) has symmetries $\hat{K}^{\chi\gamma\nu\tau} = \hat{K}^{\gamma\chi\nu\tau} = \hat{K}^{\chi\gamma\tau\nu} = \hat{K}^{\nu\tau\chi\gamma}$. Contracting either of $\eta_{\chi\gamma}\hat{K}^{\chi\gamma\nu\tau}$ (4.95) or $\eta_{\gamma\chi}\bar{K}^{\chi\nu\gamma\tau}$ (4.94) yields the same, unique, rank 2 tensor,

$$K^{\nu\tau} = 2\Box\Box\phi^{\gamma\nu\tau}_{\gamma} + 2\partial^{\nu}\partial^{\tau}\Box\phi^{\kappa\gamma}_{\kappa\gamma} + 2\partial_{\kappa}\partial_{a}\partial_{\gamma}\partial^{\tau}\phi^{\gamma\nu a\kappa} + 2\partial_{a}\partial_{\kappa}\partial_{\gamma}\partial^{\nu}\phi^{\gamma\tau a\kappa}$$
$$- 2\partial_{\kappa}\partial^{\tau}\Box\phi^{\gamma\nu\kappa}_{\gamma} - 2\partial_{\kappa}\partial^{\nu}\Box\phi^{\gamma\tau\kappa}_{\gamma} - 2\partial_{a}\partial_{\gamma}\Box\phi^{\gamma\nu\tau a} - 2\partial_{a}\partial_{\gamma}\partial^{\nu}\partial^{\tau}\phi^{\kappa\gamma a}_{\kappa}, \quad (4.96)$$

with symmetry $K^{\nu\tau} = K^{\tau\nu}$. As in the case of the standard (spin-2 curvature) Riemann tensor, we can derive a nonzero scalar curvature from this by contracting $\eta_{\tau\nu}K^{\nu\tau}$, which yields

$$K = 2\Box\Box\phi^{a\tau}_{a\tau} + 2\partial_{\kappa}\partial_{a}\partial_{\gamma}\partial_{\tau}\phi^{\kappa\tau a\gamma} - 4\partial_{\kappa}\partial_{a}\Box\phi^{\tau\kappa a}_{\tau}. \quad (4.97)$$

Therefore, in the case of spin-4, we have a Lagrangian of the form $\mathcal{L} = \mathcal{L}_8 + \mathcal{L}_6 + \mathcal{L}_4 + \mathcal{L}_2 + \mathcal{L}_0$, where the $\mathcal{L}_4$ contribution has two different possible contractions. The most general Lagrangian possible for spin-4 is then $\mathcal{L} = \tilde{D}K^{ab\gamma\chi m\kappa\nu\tau}K_{ab\gamma\chi m\kappa\nu\tau} + \tilde{E}K^{\gamma\chi m\kappa\nu\tau}K_{\gamma\chi m\kappa\nu\tau} + \Sigma_j\tilde{F}_j\mathbf{K}^{\chi\gamma\nu\tau}\mathbf{K}_{\chi\gamma\nu\tau} + \tilde{G}K^{\nu\tau}K_{\nu\tau} + \tilde{H}K^2$, where $\Sigma_j\tilde{F}_j\mathbf{K}^{\chi\gamma\nu\tau}\mathbf{K}_{\chi\gamma\nu\tau}$ represents all possible scalars built from the rank 4 curvature tensors. The 2 different curvature tensors $K^{\chi\gamma\nu\tau}$ and $\hat{K}^{\chi\gamma\nu\tau}$ present an ambiguity problem not observed in the lower spin-n models. We investigated this ambiguity by considering the most general scalar for rank 4 curvature tensors. The scalars built from each of these 2 curvature tensors are indeed solutions to the resulting linear system. Therefore we will not attempt to select one of these expressions as being superior to the others. This result shows that the curvature scalars for higher spin models can have more than one combination at each rank of curvature tensor.

Scalars built from contracted spin-n curvature tensors for $N = M = n$

We have shown by direct calculation that the curvature tensors of higher-spin gauge theories [181, 20] can be derived from the procedure in [14] without any a priori knowledge of their existence:

$$F^{[\mu\nu]} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{4.98}$$

$$R^{[\mu\nu][\alpha\beta]} = \partial^\mu\partial^\beta h^{\nu\alpha} + \partial^\nu\partial^\alpha h^{\mu\beta} - \partial^\mu\partial^\alpha h^{\nu\beta} - \partial^\nu\partial^\beta h^{\mu\alpha}, \tag{4.99}$$

$$\begin{aligned} K^{[\tau\nu][\kappa m][\chi\gamma]} = {} & \partial^\chi\partial^\tau\partial^\kappa\phi^{\gamma m\nu} + \partial^\tau\partial^m\partial^\gamma\phi^{\kappa\chi\nu} + \partial^\kappa\partial^\nu\partial^\gamma\phi^{\chi\tau m} + \partial^\chi\partial^m\partial^\nu\phi^{\kappa\tau\gamma} \\ & - \partial^\gamma\partial^m\partial^\nu\phi^{\chi\tau\kappa} - \partial^\chi\partial^\tau\partial^m\phi^{\kappa\nu\gamma} - \partial^\kappa\partial^\chi\partial^\nu\phi^{\tau m\gamma} - \partial^\tau\partial^\kappa\partial^\gamma\phi^{\chi m\nu}, \end{aligned} \tag{4.100}$$

$$\begin{aligned} K^{[ab][\gamma\chi][m\kappa][\nu\tau]} = {} & \partial^a\partial^\gamma\partial^m\partial^\nu\phi^{b\chi\kappa\tau} + \partial^a\partial^\gamma\partial^\kappa\partial^\tau\phi^{b\chi m\nu} + \partial^a\partial^m\partial^\chi\partial^\tau\phi^{b\kappa\gamma\nu} + \partial^a\partial^\nu\partial^\chi\partial^\kappa\phi^{b\tau\gamma m} \\ & + \partial^\gamma\partial^m\partial^b\partial^\tau\phi^{\chi\kappa a\nu} + \partial^\gamma\partial^\nu\partial^b\partial^\kappa\phi^{\chi\tau am} + \partial^m\partial^\nu\partial^b\partial^\chi\phi^{\kappa\tau a\gamma} + \partial^b\partial^\chi\partial^\kappa\partial^\tau\phi^{a\gamma m\nu} \\ & - \partial^a\partial^\gamma\partial^m\partial^\tau\phi^{b\chi\kappa\nu} - \partial^a\partial^\gamma\partial^\nu\partial^\kappa\phi^{b\chi\tau m} - \partial^a\partial^m\partial^\nu\partial^\chi\phi^{b\kappa\tau\gamma} - \partial^\gamma\partial^m\partial^\nu\partial^b\phi^{\chi\kappa\tau a} \\ & - \partial^a\partial^\chi\partial^\kappa\partial^\tau\phi^{b\gamma m\nu} - \partial^\gamma\partial^b\partial^\kappa\partial^\tau\phi^{\chi am\nu} - \partial^m\partial^b\partial^\chi\partial^\tau\phi^{\kappa a\gamma\nu} - \partial^\nu\partial^b\partial^\chi\partial^\kappa\phi^{\tau a\gamma m}, \end{aligned} \tag{4.101}$$

where the curvatures K have n pairs of antisymmetric indices for each spin-n that are symmetric under interchange, and the tensor potentials ϕ are all totally symmetric. Note that equations (4.98) and (4.99), the electrodynamic field strength and linearized Riemann tensor, were derived using this procedure in [14].

Since we cannot compute the Lagrangian densities built from curvature tensors of all spin-n models from the $N = M = n$ procedure, we can at best give some conjectures as to the expected form of these Lagrangian densities, based on the $n = 1$ through $n = 4$ cases. What we know is that the procedure is well defined for all n for which the gauge transformations are generalized by equations (4.77) to (4.80). Since n corresponds to the number of derivatives and the number of indices on the tensor potential, we can generalize $\mathcal{L}$ at each n using the notation $\partial_{\mu_n} = \partial_\mu \ldots \partial_\alpha$, which is a product of n partial derivatives, and $\phi_{\nu_n} = \phi_{\nu\ldots\beta}$, which is a completely symmetric tensor potential with n indices. Then the general $\mathcal{L}$ will be the sum of all possible i unique scalars of the form $(\partial_{\mu_n}\phi_{\nu_n})^2$, which we write as $\mathcal{L} = \Sigma_i C_i(\partial_{\mu_n}\phi_{\nu_n})^2$. For this generalization, we can make the following two conjectures: (i) Under a spin-n gauge transformation, a higher-spin Lagrangian density of the form $\mathcal{L} = \Sigma_i C_i(\partial_{\mu_n}\phi_{\nu_n})^2$ will have a unique gauge invariant solution that decouples into contractions of the spin-n 'Riemann' curvature tensor and its 'Ricci' tensors and scalar. This decoupling will be of the form $\mathcal{L} = \mathcal{L}_{2n} + \cdots + \mathcal{L}_4 + \mathcal{L}_2 + \mathcal{L}_0$, where $\mathcal{L}_{2n}$ is the contracted curvature tensors for the spin-n theory. Each of these curvature tensors will have n pairs of antisymmetric indices and the pairs will all be symmetric with one another. (ii) For $n = odd$, the term $\mathcal{L}_0 = 0$, leaving no 'Ricci' scalar in such models, as seen in electrodynamics and spin-3. We emphasize that in order to have an exactly gauge invariant Lagrangian—built from quadratic combinations of derivatives of potentials—for a higher spin model, one uniquely requires the contraction of independently gauge invariant 'field strength' tensors, known as the curvature tensors of higher spin gauge theories.

4.3.3 Conclusions

The curvature tensors of higher-spin gauge theories have been derived from first principles; that is, without any a priori knowledge of their existence. Using a procedure that considers the most general linear combination of scalars built from quadratic combinations of N order of derivatives and M rank of tensor potential, we explored the general case of $N = M = n$, under the spin-n gauge transformations. It had been shown in [14] that the $N = M = 1$ case uniquely determines the contraction of the field strength tensor of electrodynamics $\mathcal{L} = CF_{\mu\nu}F^{\mu\nu}$ and the $N = M = 2$ case uniquely determines the contraction of the linearized 'Riemann' and 'Ricci' tensors $\mathcal{L} = \tilde{a}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \tilde{b}R_{\mu\nu}R^{\mu\nu} + \tilde{c}R^2$. In this article we first considered the $N = M = 3$ case, under the spin-3 gauge transformation. As expected, based on the previous result, this system of linear equations decoupled into unique solutions that correspond to the contraction of the well-known curvature tensor for spin-3 and its 'Ricci' forms $\mathcal{L} = \tilde{A}K^{\tau\nu\kappa m\chi\gamma}K_{\tau\nu\kappa m\chi\gamma} + \tilde{B}K^{\tau\nu\kappa\chi}K_{\tau\nu\kappa\chi} + \tilde{C}K^{\nu\chi}K_{\nu\chi}$. This is a notable result for two reasons: (i) these curvature tensors were uniquely derived without any knowledge of their existence influencing the procedure and (ii) it provides a method for explicitly deducing these expressions, which are typically just generalized from lower rank models using inductive arguments and known symmetries (by taking the curl on each index of a totally symmetric rank-n field for each spin-n [56]). The same process was then considered for the $N = M = 4$ case, for the highest rank only, and, again, contraction of the spin-4 curvature tensor $\mathcal{L}_8 = \tilde{D}K^{ab\gamma\chi m\kappa\nu\tau}K_{ab\gamma\chi m\kappa\nu\tau}$ was uniquely determined. By considering contractions of this tensor, we wrote down the possible form of the most general Lagrangian for the $N = M = 4$ case. Finally, we provided some conjectures regarding the $N = M = n$ case.

What is interesting to note is that in [14] it was shown that, since the $N = M = n$ Lagrangians are exactly gauge invariant, one can use the Bessel-Hagen result from Noether's first theorem [26, 159] to derive gauge invariant energy-momentum tensors by fixing the remaining free coefficients in the Lagrangians. For the $N = M = 1$ case, this procedure results in the coefficients being fixed such that the Lagrangian obtained is the Lagrangian of classical electrodynamics while, for the $N = M = 2$ case, it results in the coefficients being fixed such that the Lagrangian obtained is that of the linearized Gauss-Bonnet gravity model. In this article, we obtained the result that, for higher spin models $N = M \geq 3$, if one imposes the requirement of having exactly gauge invariant Lagrangians (not merely invariant up to a surface term) built from quadratic combinations, then one again uniquely obtains the contraction of independently gauge invariant curvature tensors as a direct result of this requirement. In the past, such Lagrangians have been postulated in the physics literature but never derived [78]. We again acknowledge that the higher derivative models $N = M \geq 3$ have problems with unitary and renormalizability [2], and have no obvious predicative utility (building the models associated to these Lagrangian densities is not the purpose of our article, our purpose is to derive the the curvature tensors of higher spin gauge theories without a priori knowledge of their existence). However, having more complicated exactly gauge invariant actions can provide useful toy models to answer questions about the generalization of the Noether and Bessel-Hagen methods to more complicated theories, such as the recent use of both the $N = M = 1$ case (electrodynamics) and the $N = M = 2$ case (linearized Gauss-Bonnet gravity) in disproving the notion of general equivalence between the Noether and Hilbert energy-momentum tensors [13]. In addition the Noether identities can be used to generalize beyond the free field consideration [120]. Whether applying the Noether/Bessel-Hagen method to $N = M \geq 3$ Lagrangians will uniquely fix the free coefficients of these Lagrangians such that gauge invariant energy-momentum tensors are derived, as for $N = M = 1$ and $N = M = 2$, is the subject of future work.

Chapter 5

Conclusions

We will now summarize the 8 results that we feel are our most significant contributions to the literature from each paper (we will list the main contribution from each of our papers).

Summary of Most Significant Results

1. Application of the converse of Noether's first theorem to solve energy-momentum ambiguity problems in the literature [15]

2. Deriving several completely gauge invariant theories directly from Noether's first theorem using the Bessel-Hagen method [12]

3. Proving that the Noether and Hilbert energy-momentum tensors are not, in general, equivalent [13]

4. Proving that even for a simple scalar field model, the various energy-momentum tensors in Minkowski spacetime can diverge [10]

5. Proving that there are infinitely many spin-2 energy-momentum tensors obtainable from the ad-hoc "improvement" of the canonical Noether energy-momentum tensor, even when the Belinfante superpotential is fixed [8]

6. Developing a procedure that yields classical electrodynamics and linearized Gauss-Bonnet gravity as the $N = M = 1$ and $N = M = 2$ cases, with the variational symmetries of $h_{\mu\nu}$ being proportional to the linearized Christoffel symbol [14]

7. Developing the complete dual formulation of the $N = M = n$ case which generalizes to the Maxwell-like higher spin gauge theories [9]

8. Deriving the curvature tensors of higher spin gauge theories without relying on symmetry properties of the Riemann tensor [11]

Through these and other results in this thesis, we have outlined a concrete methodology for deriving complete sets of equations for multiple theories from a common set of axioms. By imposing axioms on a general Lagrangian density one can uniquely fix free coefficients to satisfy the axioms with respect to Noether's first theorem. The primary axioms we focused on are gauge invariance, 4D Minkowski spacetime, $N = M = n$ for spin-n gauge transformations, and conformal invariance. For a detailed outline of the motivations, and problems addressed in this thesis please see Chapter 1: Introduction. The 8 papers integrated into the body chapters, Chapter 2 [15, 12], Chapter 3 [13, 10, 8] and Chapter 4 [14, 9, 11] address the 3 problems discussed in Section 1.5 of the Introduction, respectively.

This study suggests discarding the notion that numerous distinct energy-momentum tensor should be named and symbolized ($T^{\mu\nu}$) as the same symbolized the same. A unique methodology is required for the derivation of physical conservation laws as in the case of the Euler-Lagrange equation. The Bessel-Hagen approach to Noether's first theorem provides exactly this methodology. We have shown other conflicting definitions such as the canonical Noether, Fock, Hilbert and Belinfante energy-momentum tensors diverge in certain cases and can at best be said to coincide with the physical energy-momentum tensor for some simple scalar and vector models.

What we have shown is that if one wants a unique, concrete procedure for deriving complete sets of equations for physical field theories, existing ambiguous and ad-hoc definitions of conservation laws are simply unacceptable. Only by agreeing on unique procedures for general application to physical theories can we move on to unambiguous and rational discussions about the current and future status physical field theories. In this thesis we argue that the Bessel-Hagen approach to Noether's first theorem combined with our axiomatic approach is an appropriate step in this direction.

The impact of our work on the theoretical physics literature will likely, most significantly, be the several proofs we have provided on the status of the various energy-momentum tensor definitions; these proofs are found throughout the articles in Chapter 3 [13, 10, 8]. In the past authors would calculate one of these energy-momentum tensors and assume broad equivalence to the others, such as in the aforementioned Padmanabhan-Deser debate, One can no longer be so carefree in these approaches. The debates and ambiguity issues in the past can for the most part be traced back to the absence of results which we have provided in Chapter 2.

The conventional definition of an energy-momentum tensor, the conserved current associated to the 4-parameter translation of the 10-parameter Poincare group, is clearly defined in the case of special relativistic field theory, where the Poincare group transformations are isometries of Minkowski spacetime. Beyond Minkowski spacetime serious discussion must take place as to whether or not conserved quantities which are not associated to the Poincare translation should also be given this name and symbol. Most notably this discussion should take place in the context of general relativity, where the source to Einstein's field equations are also named "energy-momentum tensor" and symbolized "$T^{\mu\nu}$". This Hilbert definition, as we have shown, conflicts with the Noether/ Poincare definition in the general setting [13].

A distinction must be made between between these conserved quantities which may lead to new insight that was hidden by the blind assumption that they are generally equivalent quantities. Such distinction, combined with our proposed application of the converse of Noether's first theorem in [15], may allow physicists to finally solve problems such as the non-uniqueness problem of energy-momentum tensors in linearized gravity.

The future goals of this line of work are numerous, so we will conclude the thesis by overviewing some of these proposed directions. The most obvious extension is to generalize the axiomatic approach in Chapter 4 to include Lagrangian densities in quantum field theory. This would require more general types of terms to be included in the general Lagrangian density with free coefficients and additional axioms to fix them. This process is more straightforward than attempting to extend these methods to general metric spacetimes where there are not finite global spacetime symmetries to apply to Noether's first theorem.

It is possible that diffeomorphism invariance can be used to fix the field equations similar to the case of linearized Gauss-Bonnet graviy and the spin-2 (linearized diffeomorphism) gauge transformation [14]. However, this is a non-exact symmetry as in the case of spin-2 Fierz-Pauli theory. The application of our methods to actions with non-exact gauge symmetries (another area of Bessel-Hagen's work) is one of our next focuses as we hope this can be applied to spin-2 to derive a unique energy-momentum tensor for linearized gravity with respect to Noether's first theorem.

Finally, one of our ultimate goals, as should be for any physicist, is to have the models we have developed tested by some sort of observation of experiment. The linearized Gauss-Bonnet gravity model which we discuss frequently throughout the thesis has not been applied in any serious sense in the literature. Recently however, the full Gauss-Bonnet gravity model has gained significant traction in the literature with new claims of predicting observed phenomena. With this renewed interest, and our results in the linearized version of the model, we hope to apply the linearized version to dynamical problems such as the galactic rotation curve problem to determine if the model that remarkably appeared out of our axioms has some practical application in predicting observed phenomena, as its electrodynamics counterpart has done so well over the past centuries.

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